Two-Generator Discrete Subgroups of Tree Automorphism Groups

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ABSTRACT

In this paper, we give a complete and explicit classification of faithful HNN 2-extensions, namely those whose actions on their Bass–Serre trees are faithful. We also construct infinite families of faithful two-generator graphs of groups realizing each admissible tree or unicyclic underlying graph. Together, these results indicate a natural threshold in the classification problem: in the small-geometry regime the groups admit an isomorphism-level classification, while beyond this barrier the natural classification is substantially coarser and is governed by quotient-graph data.

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1. Introduction

This paper studies discrete two-generator subgroups of automorphism groups of locally finite trees, especially when torsion is present. For the trivalent tree, we identify a small-geometry condition under which the group admits an isomorphism-level classification. Beyond this regime, we pass to the coarser characterization by reduced underlying graphs and construct infinite families realizing the admissible graph types.

In the classical setting, related questions arise in the study of Fuchsian groups. One such question asks when a given generating set in $\mathrm{PSL}(2,\mathbb{R})$ yields a discrete subgroup. In the two-generator case, Rosenberger [18] obtained algebraic criteria and classification results for discrete groups $\Gamma = \langle A, B\rangle$, while Gilman [11] developed a geometric treatment in terms of axes, translation lengths, and elliptic rotation angles. Another is the *constructive membership problem*: given an element of $\mathrm{PSL}(2,\mathbb{R})$, decide whether it can be expressed as a word in the given generators. Based on Rosenberger's and Gilman's work, Kirschmer and Rüther proposed an algorithm for solving the constructive membership problem for two-generator Fuchsian groups [14].

A central tool in Gilman's approach is a reduction procedure based on *Nielsen transformations*. For a hyperbolic pair with intersecting axes, the procedure transforms the generating pair either into a *Schottky pair* or into one of the *intertwining cases*. Historically, this technique originates in Nielsen's study of generating sets of free groups [17] and was subsequently extended to the theory of free products through work of Grushko [13] and Kurosh [15].

The same reduction philosophy was later applied to the non-archimedean and tree settings. For hyperbolic pairs $A, B \in \mathrm{PSL}(2, K)$ over a non-archimedean local field K, Conder introduced a similar Nielsen-move algorithm, producing either a Schottky pair or an elliptic-hyperbolic pair [4]. More recent work of Conder and Schillewaert [6] included elliptic generators as well and obtained a partial classification analogous to the classical hyperbolic case. If p is the residue characteristic of K, their classification applies under the condition that no element of Γ has order p; in particular, it is complete when $K = \mathbb{Q}_p$.

Nielsen methods were also developed directly for groups acting on simplicial trees. Weidmann [20] used Nielsen transformations to analyze finite generating sets of such groups, relating the geometry of the generators to free-product decompositions and intersection phenomena in the tree. Conder [5] generalized this Nielsen-move approach to $\mathrm{Aut}(X)$ for groups with 2 or 3 generators, and Markowitz [16] later presented the algorithm for finite generating sets in general. Our earlier work [10] focuses on the two-generator case and provides a finer characterization of when a generating pair (A, B) is discrete and free, based on geometric quantities such as the translation lengths of the generators and the length of the intersection of their axes. We also computed the corresponding geometric quantities for the resulting equivalent pair, whether it is Schottky or elliptic-hyperbolic.

As implied in [10], particular geometric quantities will result in elliptic-hyperbolic generating pairs of $\mathrm{Aut}(X)$, coming with new quantities of hyperbolic translation lengths and characteristic set intersection lengths. The following question thus natural appear:

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Question 1.1. *Let $A, B \in \mathrm{Aut}(X)$ be automorphisms of a locally finite tree X, and assume that $\Gamma = \langle A, B\rangle$ is discrete. Given the minimal displacements of A and B and the relative position of their characteristic sets, which isomorphism types can occur for Γ?*

When elliptic elements are present, the natural framework for Question 1.1 is Bass–Serre theory: after passing to the minimal Γ-invariant subtree, a finitely generated discrete subgroup is represented by a finite graph of finite groups whose fundamental group is Γ[19]. We further restrict to the case where the induced action on this minimal subtree is faithful, which is standard in, e.g.,[1].

Our study indicates that the answer depends on whether the hyperbolic translation length and the relative position quantity are small enough. Under a small-geometry assumption, the graph-of-groups decomposition of a two-generator subgroup is forced to have one of a small number of basic underlying graphs. When the quotient is an edge, the result is covered by the classical classifications of Djoković–Miller [9] and Goldschmidt [12]. The loop case instead leads us to consider faithful HNN extensions in which the edge group has index 2 in the vertex group. Accordingly, our first main result gives a classification of faithful HNN 2-extensions. The resulting family is explicit and resembles the infinite family of $(4, 2)$-amalgamated products of dihedral type [8].

Combined with our analysis of generating pairs in the amalgamated-product cases, this yields the list of two-generator discrete subgroups $\Gamma < \mathrm{Aut}(X_3)$ satisfying the following conditions:

- The generators $g, h \in \Gamma$ satisfy *small* geometric conditions: either both are elliptic with $d(T_g, T_h) = 1$, or g is elliptic and h is hyperbolic with $d(T_g, A_h) = 0$ and $\ell(h) \le 2$.
- The action of Γ is faithful on its minimal invariant subtree.

There is also a natural lattice formulation for the faithfulness condition. By Lemma 3.4, if $\Gamma < \mathrm{Aut}(X_3)$ is a lattice, then its minimal invariant subtree is X_3, so the faithfulness hypothesis above is automatic.

We show that admissible subgroups Γ either fall into finitely many amalgamated-product types or belong to a single explicit family of HNN extensions. In addition, when the generator g is elliptic, its order is restricted to $2, 3, 4, 6, 8$, or 12.

The preceding classification reaches a natural threshold. Once these small-geometry restrictions are relaxed—for example, when $\ell(h) > 2$, $d(T_g, A_h) \ge 1$, or the ambient tree X has vertices of valency at least 4—higher-index amalgams such as $(4, 3)$- and $(6, 2)$-amalgams arise, and the low-index amalgam classifications no longer apply. We therefore turn to a coarser level of classification, describing these groups by the reduced underlying graphs of their graph-of-groups decompositions.

For one direction, we determine the conditions under which a finite graph can occur as the reduced underlying graph of a discrete subgroup with prescribed geometric quantities. In the special case $d(T_g, A_h) = 0$, this criterion can be written explicitly. The converse direction is constructive: for every admissible quotient graph, we explicitly construct faithful two-generator graphs of groups realizing the prescribed translation length $\ell(h)$ and distance $d(T_g, A_h)$. These constructions form infinite families with vertex stabilizers of arbitrarily large order.

The paper is organized as follows. In Section 2, we review the Bass–Serre theoretic notions used throughout the paper. In Section 3, we present our first set of results concerning rigidity for two-generator discrete subgroups of $\mathrm{Aut}(X_3)$ with small geometric quantities. In Section 4, we derive restrictions on the minimal quotient graphs of general two-generator subgroups. Explicit constructions of infinite families of groups are given in Section 5, and in Section 6 we use these constructions to describe the possible reduced graphs.

2. Preliminaries: Graphs of Groups

Throughout this paper, we assume that X is a locally finite tree and that the quotient $X/\mathrm{Aut}(X)$ is a finite graph. Let $\mathcal{V}(X)$ and $\mathcal{E}(X)$ denote the sets of its vertices and edges, respectively.

A tree automorphism is either an *inversion*, an *elliptic* automorphism, or a *hyperbolic* automorphism. If g is an elliptic automorphism, its fixed subset $T_g \subset X$ is a subtree of X. If h is hyperbolic, then the displacement $d(v, h \cdot v)$ is always strictly positive; it attains a minimum $\ell(h)$, called its *translation length*, precisely on a subset $A_h \subset X$ called its *axis*, which is a bi-infinite geodesic in X [19].

The total automorphism group $\mathrm{Aut}(X)$ is equipped with a topology in which the pointwise stabilizers of finite induced subtrees form a basis of open neighborhoods of $\mathrm{Id}_{\mathrm{Aut}(X)}$. This leads to the notion of *discreteness* for $\mathrm{Aut}(X)$-subgroups:

Definition 2.1. *A subgroup $G < \mathrm{Aut}(X)$ is* discrete *if there exists a finite tree $T \subset X$ such that the pointwise stabilizer $G_T = 1$.*

In particular, if $G < \mathrm{Aut}(X)$ is discrete and finitely generated, then all stabilizer subgroups G_v and G_e are finite, and their orders are uniformly bounded.

Unless otherwise specified, whenever we mention a subgroup $G < \mathrm{Aut}(X)$ in this paper, we assume that G is discrete, finitely generated, and inversion-free.

To study the action of a tree automorphism group in an algebraic framework, one introduces the notion of a *graph of groups* [7]:

Definition 2.2. *A* ***graph of groups*** *$(Y, \mathcal{G})$ consists of the following data:*

- *A connected and symmetric directed graph Y with vertex set $\mathcal{V}(Y)$ and edge set $\mathcal{E}(Y)$.*
- *A family $\mathcal{G}$ of groups, including* vertex groups *$G(v)$ for $v \in \mathcal{V}(Y)$, and* edge groups *$G(e)$ for $e \in \mathcal{E}(Y)$.*
- *The* boundary monomorphisms

$$\iota_{e,\alpha} : G(e) \to G(\alpha_e),\ \iota_{e,\omega} : G(e) \to G(\omega_e)$$

for each edge $e \in \mathcal{E}(Y)$ and the incident vertices α_e and ω_e.

We regard a graph of groups as a combinatorial generalization of hyperbolic orbifolds. The *fundamental group* of a graph of groups is defined analogously:

Definition 2.3. *Let $(Y, \mathcal{G})$ be a graph of groups, and fix a spanning tree $Y_0 \subset Y$. Its* fundamental group *(or* amalgam*) $\pi_1(Y, \mathcal{G}; Y_0)$ is defined as*

$$\pi_1(Y, \mathcal{G}; Y_0) = \left(\left(*_{v \in \mathcal{V}(Y)} G(v) \right) * F_{\mathcal{E}(Y) \setminus \mathcal{E}(Y_0)} \right) / \sim,$$

where $F_{\mathcal{E}(Y)\setminus\mathcal{E}(Y_0)}$ is the free group generated by symbols t_e corresponding to (oriented) edges $e \in \mathcal{E}(Y) \setminus \mathcal{E}(Y_0)$, subject to the relations:

$$\iota_{e,\omega}(g) = t_e \iota_{e,\alpha}(g) t_e^{-1}.$$

One shows that this definition is independent of the choice of the spanning tree Y_0, up to isomorphism. Thus, the fundamental group is also denoted by $\pi_1(Y, \mathcal{G})$.

In analogy with the manifold setting, one considers the *universal covering* (or *Bass-Serre tree*) and *fundamental group* of a graph of groups.

Definition 2.4. *Let $(Y, \mathcal{G})$ be a graph of groups, and denote its fundamental group by $\pi_1(Y, \mathcal{G})$. Define a graph $\widetilde{X} = \widetilde{X}(Y, \mathcal{G})$, called the* Bass-Serre covering tree*, whose vertices and edges are cosets:*

$$\mathcal{V}(\widetilde{X}) = \{ gG(v) \mid g \in \pi_1(Y, \mathcal{G}), v \in \mathcal{V}(Y) \},$$

and

$$\mathcal{E}(\widetilde{X}) = \{ gG(e) \mid g \in \pi_1(Y, \mathcal{G}), e \in \mathcal{E}(Y) \}.$$

Moreover, the incidence is determined as follows:

- *If $e \in \mathcal{E}(Y_0)$,*

$$\alpha_{gG(e)} = gG(\alpha_e),\ \omega_{gG(e)} = gG(\omega_e).$$

- *If $e \notin \mathcal{E}(Y_0)$,*

$$\alpha_{gG(e)} = gG(\alpha_e),\ \omega_{gG(e)} = (gt_e)G(\omega_e).$$

The resulting graph $\widetilde{X}$ is a tree, and is independent of the choice of the spanning tree up to isomorphism. The group $\pi_1(Y, \mathcal{G})$ acts on $\widetilde{X}$ by left multiplication on cosets.

Definition 2.5. *Let X be a tree, and $\Gamma < \mathrm{Aut}(X)$ a discrete subgroup. We say that a subset $\mathcal{Y}$ of vertices and edges, together with a family $\mathcal{G}$ of groups, is a* fundamental domain *for Γ if the following conditions hold:*

- *The set $\mathcal{Y}$ is Γ-transversal, meaning that $\mathcal{Y}$ meets each Γ-orbit of vertices and each Γ-orbit of edges exactly once.*
- *The maximal subtree $Y_0 \subset \mathcal{Y}$ contains every vertex in $\mathcal{V}(\mathcal{Y})$ and the origin of every edge in $\mathcal{E}(\mathcal{Y})$.*
- *The vertex groups $G(v)$ and edge groups $G(e)$ are the stabilizers Γ_v and Γ_e for $v \in \mathcal{V}(\mathcal{Y})$ and $e \in \mathcal{E}(\mathcal{Y})$.*

Proposition 2.1. *Let X be a tree, $\Gamma < \mathrm{Aut}(X)$ be a discrete subgroup, and $(\mathcal{Y}, \mathcal{G})$ be a fundamental domain for Γ. By identifying the unpaired termini of $\mathcal{Y}$-edges with the $\mathcal{Y}$-vertices of the same Γ-orbit, we identify $\mathcal{Y}$ with the quotient graph $Y = X/\Gamma$ and regard $(\mathcal{Y}, \mathcal{G})$ as a graph of groups $(Y, \mathcal{G})$. Moreover:*

- *The fundamental group $\pi_1(Y, \mathcal{G})$ is isomorphic to Γ.*
- *The natural inclusion $j : \mathcal{Y} \hookrightarrow X$ extends to an isomorphism from the Bass-Serre covering tree $\widetilde{X}(Y, \mathcal{G})$ to X.*

The following examples from [19] are key constructions of graphs of groups used in this paper.

Example 2.1 (Amalgamated Products)**.** *Let Y be a line segment, with vertices v^+, v^- and edge e. Define a graph of groups $(Y, \mathcal{G})$ such that $G_1 = \mathcal{G}(v^+)$, $G_2 = \mathcal{G}(v^-)$, and $H = \mathcal{G}(e)$, where $H < G_1$ and $\iota_{e,v^-} = \varphi : H \hookrightarrow G_2$ is an injective homomorphism. Then the fundamental group,*

$$\pi_1(Y, \mathcal{G}) \cong G_1 *_H G_2 := (G_1 * G_2)/(h \sim \varphi(h),\ h \in H)$$

is an amalgamated free product *of G_1 and G_2 over H.*

If both $|G_1 : H| = n$ and $|G_2 : H| = m$ are finite, then the Bass-Serre covering tree of $(Y, \mathcal{G})$ is an (n, m)-biregular tree.

Example 2.2 (HNN Extensions)**.** *Let Y be a loop, with vertex v and edge e. Define a graph of groups $(Y, \mathcal{G})$ such that $G = \mathcal{G}(v)$ and $H = \mathcal{G}(e)$. Both endpoints of e are equal to v, so we can let $H < G$ and $\iota_{e,\alpha}$ be the identity. Let $\iota_{e,\omega} = \varphi : H \hookrightarrow G$. Then the fundamental group*

$$\pi_1(Y, \mathcal{G}) \cong G *_\varphi := (G * \langle x \rangle)/(\varphi(h) \sim xhx^{-1},\ h \in H),$$

is called an HNN extension *of G. Here, x is a symbol not in G of infinite order, called the* stable letter.

If $|G : H| = n$ is finite, then the Bass-Serre covering tree of $(Y, \mathcal{G})$ is a $2n$-regular tree. The stable letter x acts on the covering tree as a hyperbolic automorphism of translation length $\ell(x) = 1$.

If the quotient $Y = X/\Gamma$ by a subgroup $\Gamma < \mathrm{Aut}(X)$ is infinite, it is more convenient to consider its action on a smaller subtree:

Definition 2.6 ([7])**.** *Let $\Gamma < \mathrm{Aut}(X)$ be a subgroup containing a hyperbolic element. The union of all axes A_g of hyperbolic elements $g \in \Gamma$ is a subtree of X, called the minimal Γ-subtree of X.*

The minimal Γ-subtree T is the unique minimal subtree preserved by Γ. Moreover, if Γ is finitely generated, the quotient T/Γ is a finite graph.

We call $Y_0 = T/\Gamma$ the *minimal quotient graph* for Γ, and the fundamental domain (Y_0, Γ) (as in Definition 2.5) the *minimal quotient graph of groups* for Γ.

Intuitively, the minimal Γ-subtree is obtained by shrinking all subtrees of X on which Γ acts trivially relative to their root vertices. The boundary monomorphisms within such subtrees are bijective and can be reduced without changing the fundamental group. Bijective boundary monomorphisms may persist in the minimal quotient graph of groups of Γ, suggesting that further reductions are possible:

Definition 2.7. *Let e be a non-loop edge in a graph of groups $(Y, \mathcal{G})$ and let v_1, v_2 be the incident vertices. If the edge group $\mathcal{G}(e)$ is isomorphic to the vertex group $\mathcal{G}(v_1)$, we can reduce the graph of groups as follows:*

- *Shrink the edge e and merge vertices v_1 and v_2 into a single vertex v, with $\mathcal{G}(v) = \mathcal{G}(v_2)$.*
- *The edges incident to v are those that were incident to either v_1 or v_2 before the reduction (excluding e itself).*
- *For edges incident to v_1, the new boundary monomorphism is $\iota_{e',v} = \iota_{e,v} \circ \iota_{e,v_1}^{-1} \circ \iota_{e',v_1}$. For edges incident to v_2, the new boundary monomorphism is $\iota_{e',v} = \iota_{e',v_2}$.*

The fundamental group $\pi_1(Y, \mathcal{G})$ remains unchanged after such a reduction. A graph of groups $(Y, \mathcal{G})$ is called reduced *if it admits no further reductions—that is, if each boundary monomorphism is proper.*

For $\Gamma < \mathrm{Aut}(X)$ with a minimal quotient graph of groups (Y_0, Γ), any reduction $(Y, \mathcal{G})$ of (Y_0, Γ) that is itself reduced is called a *reduced graph of groups* associated with Γ. The underlying graph Y is called a *reduced graph* associated with Γ. When the minimal fundamental domain $(Y, \mathcal{G})$ of a subgroup $\Gamma < \mathrm{Aut}(X)$ is itself reduced, we simply say that the quotient graph $Y = T/\Gamma$ is a *reduced quotient* of Γ when it is unambiguous.

The criterion for faithfulness of the fundamental-group action on the Bass-Serre covering tree follows naturally from the definition:

Proposition 2.2. *Let $(Y, \mathcal{G})$ be a graph of groups, let $X = \widetilde{X}(Y, \mathcal{G})$ be its Bass-Serre covering tree, and let $\Gamma = \pi_1(Y, \mathcal{G})$ be its fundamental group. Then the canonical action of Γ on X is faithful if and only if the following kernel is trivial for every vertex $e \in \mathcal{E}(Y)$:*

$$\bigcap_{\gamma \in \Gamma} \gamma \mathcal{G}_e \gamma^{-1} = \{1\}.$$

A graph of groups $(Y, \mathcal{G})$ whose fundamental group does not act faithfully on its covering tree $\widetilde{X} = \widetilde{X}(Y, \mathcal{G})$ cannot be realized as a subgroup of $\mathrm{Aut}(\widetilde{X})$. Nevertheless, we can understand it as an automorphism subgroup of a larger tree by adding a descending chain of subgroups. As shown in the following example, this procedure explains the significance of requiring a faithful action on the minimal tree in the subsequent sections.

Example 2.3 (Finite-by-D_∞ groups acting non-faithfully on the minimal subtree)**.** *Let K be a group with $|K| = 2^k$, $k > 0$. Let G and H be extensions of K, each of order 2^{k+1}. Every 2-group admits a descending chain of subgroups:*

$$G = G_{k+1} \rhd G_k \rhd \cdots \rhd G_1 \rhd G_0 = \{1\}, \quad H = H_{k+1} \rhd H_k \rhd \cdots \rhd H_1 \rhd H_0 = \{1\}, \quad |G_i| = |H_i| = 2^i.$$

Consider an amalgamated product:

$$\Gamma = G_0 *_{G_0} G_1 *_{G_1} \cdots *_{G_k} G *_G G *_K H *_H H *_{H_k} \cdots *_{H_1} H_1 *_{H_0} H_0.$$

Then Γ is the fundamental group of a path of groups. From the edge-vertex indices, its Bass-Serre universal covering tree is contained in a trivalent tree, and the existence of trivial edge groups G_0 and H_0 implies that the action on X_3 is faithful. Therefore, Γ is a discrete subgroup of $\mathrm{Aut}(X_3)$.

On the other hand, the edge reduction of the aforementioned path of groups yields a path of unit length, corresponding to the fact that $\Gamma \cong G *_K H$. Consequently, the minimal Γ-subtree is a bi-infinite path, and K is the kernel of the action of Γ restricted to this minimal subtree. The group Γ is isomorphic to a semidirect product $K \rtimes_\varphi D_\infty$ for some $\varphi : D_\infty \to \mathrm{Out}(K)$.

Following a similar argument, one sees that the HNN extension

$$\Gamma = G_0 *_{G_0} G_1 *_{G_1} \cdots *_{G_k} G *_G G *_\varphi, \quad |G| = 2^{k+1}, \quad \varphi : \mathbb{Z} \to \mathrm{Out}(G),$$

is isomorphic to the finite-by-$\mathbb{Z}$ group $G \rtimes_\varphi \mathbb{Z}$ and is also a non-faithful discrete subgroup of $\mathrm{Aut}(X_3)$.

3. Partial Classification of Two-generator Discrete Subgroups of $\mathrm{Aut}(X_3)$

This section addresses our first goal in this paper: classifying the isomorphism types of discrete subgroups of $\mathrm{Aut}(X_3)$ under certain restrictions. In particular, the small-geometry condition guarantees a graph-of-groups description with a bounded number of edges. When the graph is an edge with two endpoints, the faithfulness condition allows classical results by Djoković-Miller [9] and Goldschmidt [12] to play an important role. By contrast, when the graph is a loop, our main objective requires a classification of faithful 2-HNN extensions, which is detailed in the subsection below.

We note that this 2-HNN extension result also yields a classification for the $(2, 4)$-biregular tree $X_{2,4}$ with the same small-geometry and faithfulness restrictions.

3.1. Faithful 2-HNN Extensions

Recall from Proposition 2.2 that an HNN extension $G *_\varphi$ is faithful if

$$\cap_{n\in\mathbb{Z}} x^n G x^{-n} = \{1\},$$

where x denotes the stable letter. The main result of this subsection is a classification of faithful 2-HNN extensions into a single infinite family.

Theorem 3.1. *Suppose G is a finite group, $H < G$ is a subgroup with $|G : H| = 2$, $\varphi : H \to G$ is an injection, and the HNN extension $\Gamma = G *_\varphi$ is faithful. Then there exist parameters $n \geq 1$, $0 \leq k \leq \lfloor \frac{n}{3} \rfloor$, and $\epsilon(j,l) \in \{0,1\}$ for $n-k \leq j \leq n-1$, $k \leq l \leq j-k$, such that $\sum_l \epsilon(n-k,l) > 0$, and G is isomorphic to $G_{n,k,\vec{\epsilon}}$ with the following presentation:*

$$G_{n,k,\vec{\epsilon}} = \langle t_1, \dots, t_n \mid t_i^2, [t_i, t_{i+j}],\ 0 < j < n-k,\ [t_i, t_{i+j}] t_{i+k}^{\epsilon(j,k)} \dots t_{i+j-k}^{\epsilon(j,j-k)},\ n-k \leq j \leq n-1 \rangle.$$

In addition, the subgroup H is identified with the subgroup $\langle t_1, \dots, t_{n-1} \rangle$, and φ is given by $\varphi(t_i) = t_{i+1}$ for $i = 1, \dots, n-1$. Therefore $\Gamma \cong \Gamma_{n,k,\vec{\epsilon}} := G_{n,k,\vec{\epsilon}} *_\varphi$.

Remark 3.1. *Djoković [8] studied faithful $(4,2)$-amalgams; besides a finite number of amalgams of A_4- and S_4-type, he showed that the remainder is an infinite family of D_8-type amalgams. We note that the vertex stabilizers of the HNN extensions in our classification are described in a manner very similar to the edge stabilizers for those D_8-type $(4,2)$-amalgams.*

Before proving the theorem, we observe a few examples.

Example 3.1. *As a basic example, when $n = 1$ and $k = 0$, there are no parameters $\epsilon(j,l)$, and we have $G = C_2$, $H = \{1\}$, and $\Gamma = C_2 * \mathbb{Z}$. This occurs as a two-generator discrete subgroup of $\mathrm{PSL}(2, \mathbb{Q}_2)$ in [6], Theorem A.*

Example 3.2. *Let $n \geq 3$, $k = 1$, $\epsilon(n-1,1) = 1$, and $\epsilon(n-1,l) = 0$ for $l > 1$. Then $G \cong D_8 \times (C_2)^{n-3}$ and $H_1, H_2 \cong (C_2)^{n-1}$. The HNN extension Γ is generated by the stable letter $h = x$ and an elliptic automorphism $g = t_1 t_{n-1}$. Although the vertex-group orders are arbitrarily large in this subfamily, the generating pair (g,h) satisfies a uniform geometric condition, namely $\mathrm{ord}(g) = 2$ and $\ell(T_g \cap A_h) = 1$.*

To proceed to the proof of Theorem 3.1, we begin with a few lemmas:

Lemma 3.1. *The group G satisfying the requirements in Theorem 3.1 is a 2-group. In addition, if $|G| = 2^n$, then G is generated by n involutions $t_1, \dots, t_n$, for which the HNN isomorphism satisfies $\varphi(t_i) = t_{i+1}$.*

Proof. Suppose a group G together with subgroups H_1, H_2 of index 2 and an isomorphism $\varphi : H_1 \to H_2$ defines a faithful HNN extension $\Gamma := G *_\varphi$. Then Γ is the fundamental group of the 2-circuit whose vertex groups and one edge group are G, while the other edge group is isomorphic to H_1; the map φ corresponds to the HNN stable letter. Primitivity requires that G be a finite $\mathrm{Aut}(X_3)$-edge stabilizer subgroup, so G is a 2-group.

Inductively define

$$K_0 = G, \quad K_i = \varphi(H_1 \cap K_{i-1}).$$

Then $K_{i+1} \leq K_i$, and

$$|G : K_{i+1}| \leq |G : H_1| \cdot |G : K_i| = 2|G : K_i|.$$

If $K_{i+1} = K_i \neq \{1\}$, then φ preserves a non-trivial subgroup of G, contradicting primitivity. Therefore $K_{i+1} < K_i$ with index 2. If $|G| = 2^n$, then

$$\{1\} = K_n < K_{n-1} < \cdots < K_1 = H_2 < K_0 = G.$$

Let t_n be the generator of K_{n-1} of order 2. For $i = 1, \dots, n-1$, define $t_i = \varphi^{i-n}(t_n)$. It follows that

$$t_i \in K_{i-1},\ t_i \notin K_i,\ i = 1, \dots, n,$$

implying $|\langle t_1, \dots, t_n \rangle| \geq 2^n = |G|$. Since $t_1, \dots, t_n \in G$, we conclude that G admits the generating set $(t_1, \dots, t_n)$ with $\varphi(t_i) = t_{i+1}$.

□

From the argument above, we see that

$$K_i = \langle t_{i+1}, \dots, t_n \rangle$$

is a subgroup of G of order 2^{n-i}. For $0 \le i < j \le n$, set

$$K_{i,j} = \langle t_{i+1}, \dots, t_j \rangle.$$

Then $K_{i,j} = \varphi^{j-n}(K_{i-j+n})$ is a subgroup of order 2^{j-i}.

Lemma 3.2. *Every element of G has a unique expression as a product*

$$g = t_{i_1} \dots t_{i_k},\ 1 \le i_1 < \dots < i_k \le n.$$

Moreover, the group G admits commutator relations

$$[t_i, t_{i+l}] = t_{i+1}^{\epsilon(l,1)} t_{i+2}^{\epsilon(l,2)} \dots t_{i+l-1}^{\epsilon(l,l-1)},$$

for $\epsilon(l,1),\dots, \epsilon(l,l-1) \in \{0,1\}$.

Proof. First, we show that for any $i < j$, the commutator

$$[t_i, t_j] \in K_{i,j-1}.$$

Indeed, $K_{i,j-1}$ is a normal subgroup of index 2 in both of the following:

$$K_{i,j-1} \lhd K_{i-1,j-1}, \text{ and } K_{i,j-1} \lhd K_{i,j}.$$

Since $K_{i-1,j-1}K_{i,j} = K_{i-1,j}$, $K_{i,j-1}$ is also normal in $K_{i-1,j}$. Consider the quotient groups:

$$K_{i-1,j-1}/K_{i,j-1}, \quad K_{i,j}/K_{i,j-1} < K_{i-1,j}/K_{i,j-1}.$$

The first two are isomorphic to C_2 and generated by $[t_i]$ and $[t_j]$; these elements generate the latter group, which is of order 4. Therefore $[t_i]$ and $[t_j]$ commute in $K_{i-1,j}/K_{i,j-1}$, so $[t_i, t_j] \in K_{i,j-1}$.

This fact implies that $[t_i, t_j]$ is a product of certain letters from t_{i+1} through t_{j-1}. Since each t_i is an involution, this allows each element $g \in G$ to be expressed as a product of t_i's with increasing indices. Since there are 2^n such words and $|G| = 2^n$, the expressions are unique. Such expressions allow the relators to take the form described in the lemma. □

Proof of Theorem 3.1. We will prove by induction that the relators of such a group satisfy the presentation in Theorem 3.1. The base case is clear: for orders 2^1 and 2^2, $G = C_2$ and $G = V_4$ are the only eligible groups.

For convenience, we assume that the theorem holds for groups of order 2^{n-1}, and we prove the case $|G| = 2^n$ under this assumption. By Lemma 3.1, the subgroups $H_1 = \langle t_1, \dots, t_{n-1} \rangle$ and $H_2 = \langle t_2, \dots, t_n \rangle < G$ satisfy the hypotheses of the theorem. Thus, we can assume that there exists $0 \le k_0 \le \frac{n-1}{3}$ such that t_i and t_{i+j} commute for any $j \le n-2-k_0$, and for $n-1-k_0 \le l \le n-2$,

$$[t_i, t_{i+l}] = t_{i+k_0}^{\epsilon(l,k_0)} \dots t_{i+l-k_0}^{\epsilon(l,l-k_0)},$$

where $\sum_j \epsilon(n-1-k_0, j) > 0$. Set

$$k_+ = \max_l(\min_{\epsilon(l,k)=1} k),\ k_- = \max_l(l - \max_{\epsilon(l,k)=1} k).$$

The induction hypothesis implies that $k_+, k_- \ge k_0$, while the case $|G| = 2^n$ further requires that $k_+, k_- \ge k_0 + 1$.

Suppose that $k_+ = k_0$; then there exists $l_0 \ge n-1-k_0$ such that

$$(t_1 t_{l_0+1})^2 = t_{k'_1} \dots t_{k'_j},$$

with $k_0 + 1 = k'_1 \leq \cdots \leq k'_j \leq l - k_0 + 1$.

Now, since H_1 is a normal subgroup of G, consider the automorphism

$$\psi : H_1 \to H_1,\ t \mapsto t_n t t_n.$$

The induction hypothesis implies that $\psi(t_j) = t_j$ for $j \geq k_0 + 2$, and

$$\psi(t_j) = t_j t_{j+k_0}^{\epsilon(n-j,k_0)} \dots t_{n-k_0}^{\epsilon(n-j,n-j-k_0)},$$

for $j = 2, \dots, k_0 + 1$. In particular, the fact that $\sum_l \epsilon(n-1-k_0, l) > 0$ in the definition of k_0 implies that $[t_{k_0+1}, t_n]$ is non-trivial; equivalently,

$$\psi(t_{k_0+1}) = t_{k_0+1} t_{k_1} \dots t_{k_i} \neq t_{k_0+1},$$

with $2k_0 + 1 \leq k_1 < \cdots < k_i \leq n - k_0$ and $i \geq 1$.

Combining the facts above, we obtain:

$$\begin{aligned}
&(\psi(t_1)t_{l_0+1})^2 = (\psi(t_1)\psi(t_{l_0+1}))^2 = \psi((t_1 t_{l_0+1})^2)\\
=&\psi(t_{k_0+1} t_{k'_2} \dots t_{k'_j}) = \psi(t_{k_0+1}) t_{k'_2} \dots t_{k'_j}\\
\neq& t_{k_0+1} t_{k'_2} \dots t_{k'_j} = (t_1 t_{l_0+1})^2.
\end{aligned}$$

While the induction hypothesis does not allow $\psi(t_1)$ to fit into the pattern, Lemma 3.2 guarantees that $t_1\psi(t_1)$ can be expressed as a word in the letters $t_2, \dots, t_{n-1}$ in ascending order. If $t_1\psi(t_1)$ is a word involving only the letters $t_{k_0+1}, \dots, t_{n-1}$, then the commutativity of t_{l_0+1} with $t_1\psi(t_1)$ implies that $(\psi(t_1)t_{l_0+1})^2 = (t_1 t_{l_0+1})^2$, a contradiction. Therefore, if we let

$$m = \min_{\epsilon(n-1,k)=1} k,$$

then $m \leq k_0 - 1$, that is, $m + n - k_0 \leq n - 1$. The commutator $[t_1, t_{m+n-k_0}]$ can be expressed as a word in the letters $t_{k_0+1}, \dots, t_{n-k_0-1}$.

If this word expression of $[t_1, t_{m+n-k_0}]$ does not contain t_{k_0+1}, then it is fixed by the automorphism ψ. The element t_{m+n-k_0} is also in this fixed-point subgroup:

$$[t_1, t_{m+n-k_0}] = \psi([t_1, t_{m+n-k_0}]) = [\psi(t_1), \psi(t_{m+n-k_0})] = [\psi(t_1), t_{m+n-k_0}].$$

By assumption, $\psi(t_1) = t_1 t_{m+1} \prod_{j \geq m+1} t_{j+1}^{\epsilon(n-1,j)}$, and t_{m+n-k_0} commutes with every such t_{j+1} as $j + 1 \geq m + 2$. As a result,

$$[t_{m+1}, t_{m+n-k_0}] = 1,$$

contradicting the fact that $\sum_l \epsilon(n-1-k_0, l) > 0$.

If t_{k_0+1} appears in the commutator $[t_1, t_{m+n-k_0}]$, we need a further lemma:

Lemma 3.3. *In the setting above, we have $\epsilon(n - k_0 - 1, n - 2k_0 - 1) = 0$. That is, t_{n-k_0} does not appear as a letter in the word expression of $\psi(t_{k_0+1})$.*

Proof. Since t_1 commutes with t_{k_0+1}, we have

$$[\psi(t_1), \psi(t_{k_0+1})] = \psi([t_1, t_{k_0+1}]) = 1.$$

If t_{n-k_0} appears in the word expression of $\psi(t_{k_0+1})$, then any other factor lies between t_{k_0+1} and t_{n-k_0-1}, hence lies in the center of $H_1 = \langle t_1, \dots, t_{n-1} \rangle$ and therefore commutes with $\psi(t_1)$. Therefore,

$$[\psi(t_1), t_{n-k_0}] = 1.$$

However, every factor in $\psi(t_1)$ except for t_1 itself commutes with t_{n-k_0}, implying $[t_1, t_{n-k_0}] = 1$, which contradicts the definition of k_0.

□

Returning to the proof of Theorem 3.1. Since t_{k_0+1} appears in $[t_1, t_{m+n-k_0}]$, a slightly different relation holds:

$$[\psi(t_1), t_{m+n-k_0}] = \psi([t_1, t_{m+n-k_0}]) = t_{k_0+1}\psi(t_{k_0+1})[t_1, t_{m+n-k_0}].$$

As in the previous case, the fact that t_{m+n-k_0} commutes with every factor in $\psi(t_1)$ except for t_1 and t_{m+1} implies

$$[t_{m+1}, t_{m+n-k_0}] = t_1 t_{k_0+1}\psi(t_{k_0+1})t_1.$$

However, Lemma 3.3 implies that the index of each letter in the word expression of $\psi(t_{k_0+1})$ is strictly less than $n-k_0$, so $\psi(t_{k_0+1})$ commutes with t_1. Hence,

$$[t_{m+1}, t_{m+n-k_0}] = t_{k_0+1}\psi(t_{k_0+1}) = [t_{k_0+1}, t_n],$$

or

$$\prod_{j=k_0}^{n-2k_0-1} t_{m+1+j}^{\epsilon(n-k_0-1,j)} = \prod_{j=k_0}^{n-2k_0-1} t_{k_0+1+j}^{\epsilon(n-k_0-1,j)},$$

which contradicts the definition of k_0, the fact that $m < k_0$, and the unique ascending-letter expression of elements in G. This shows that $k_+ \geq k_0 + 1$. A similar argument for the t_1-conjugation automorphism on H_2 implies that $k_- \geq k_0 + 1$.

It remains to show that the commutator $[t_1, t_n] = (t_1 t_n)^2$ also fits the pattern of the other commutators; that is, it is a word in ascending letters among t_{k_0+2} through t_{n-k_0-1}. As mentioned earlier, Lemma 3.2 implies that $[t_1, t_n]$ is a word in letters from t_2 to t_{n-1}.

First, suppose the word expression of $[t_1, t_n]$ contains letters from t_2 to t_{k_0}. In the definition of $m = \min_{\epsilon(n-1,k)=1} k$ above, this implies $m \leq k_0 - 1$. The fact that $k_+ \geq k_0 + 1$ implies that the word expression of $[t_1, t_{m+n-k_0}]$ does not contain the letter t_{k_0+1}, leading to the same contradiction, namely that $[t_{m+1}, t_{m+n-k_0}] = 1$. Therefore, $[t_1, t_n]$ does not contain letters from t_2 to t_{k_0}, and similarly, it does not contain letters from t_{n-k_0+1} to t_{n-1}.

Now suppose $[t_1, t_n]$ contains the letter t_{k_0+1}, that is,

$$\psi(t_1) = t_1 t_{k_0+1} w,$$

for some word w in letters with indices $\geq k_0 + 2$ and $< n$. Since conjugation by t_n, namely ψ, has order 2, we have

$$t_1 = \psi^2(t_1) = \psi(t_1)\psi(t_{k_0+1})\psi(w) = t_1 t_{k_0+1} w t_{k_0+1} w' w,$$

where $w' = t_{k_0+1}\psi(t_{k_0+1})$ is non-identity by the definition of k_0. Now, the fact that the word w has indices between $k_0 + 2$ and $n-1$ implies that $w^2 = 1$ and $[w, t_{k_0+1}] = 1$, so

$$1 = t_{k_0+1} w t_{k_0+1} w' w = w w' w \implies w' = w^2 = 1,$$

a contradiction. Similarly, t_{n-k_0} is not in the word expression of $[t_1, t_n]$, proving our claim.

□

To close this subsection, let us discuss the order of elliptic elements in faithful 2-HNN extensions Γ. Every elliptic element in $\Gamma_{n,k,\vec{\epsilon}}$ has order at most 4: every element in G is contained in the abelian subgroup $\langle t_{k+1}, \ldots, t_{n-k}\rangle$, and the fourth power equals the identity. In addition, $\Gamma_{n,k,\vec{\epsilon}}$ contains an elliptic element of order 4 if and only if $G_{n,k,\vec{\epsilon}}$ is non-abelian, that is, if $k > 0$.

The existence of a generating pair (g, h) of Γ with $\mathrm{ord}(g) = 2$ is straightforward: take $g = t_1$ and $h = x$, so x is hyperbolic, t_1 is elliptic of order 2, and $\Gamma = \langle t_1, x\rangle$. On the other hand, for $k > 0$, a generating pair (g, h) with $\mathrm{ord}(g) = 4$ is guaranteed by the proposition below.

Proposition 3.1. *Let $\Gamma = \Gamma_{n,k,\vec{\epsilon}} = G_{n,k,\vec{\epsilon}} *_\varphi$ as defined earlier, where $k > 0$ and $\epsilon(n-k, l) = 1$ for some $k \leq l \leq n-2k$. Then Γ is generated by the stable letter x and an elliptic element $g \in G$, with* $\mathrm{ord}(g) = 4$.

Proof. Using the same notation as in Theorem 3.1, suppose that

$$[t_1, t_{1+n-k}] = t_{1+k_1} \cdots t_{1+k_m},$$

where $m \geq 1$ and $k \leq k_1 < \cdots < k_m \leq n - 2k$. Since $[t_i, t_{i+j}] = 1$ for any $j < n - k$, it follows that

$$(t_1 t_{1+k_2-k_1} \cdots t_{1+k_m-k_1} t_{1+n-k})^2 = (t_1 t_{1+n-k})^2 = t_{1+k_1} \cdots t_{1+k_m}.$$

On the other hand,

$$\varphi^{-k_1}(t_{1+k_1} \cdots t_{1+k_m}) = t_1 t_{1+k_2-k_1} \cdots t_{1+k_m-k_1}.$$

Therefore, the stable letter x, together with $g = t_1 t_{1+k_2-k_1} \cdots t_{1+k_m-k_1} t_{1+n-k}$, generates Γ. The equations above imply that $\mathrm{ord}(g) = 4$.

□

3.2. Two-generator $(3, 2)$- and $(3, 3)$-amalgamated products

To complete the classification envisioned above for two-generator discrete subgroups of $\mathrm{Aut}(X_3)$ with small geometric quantities, we still need to analyze the faithful $(3, 2)$- and $(3, 3)$-amalgamated products. Classical results by Djoković-Miller [9] and Goldschmidt [12] suggest that there are only finitely many of them:

Theorem 3.2. *Let $\Gamma = G_1 *_H G_2$ be a faithful amalgamated product of finite groups with $|G_1 : H|, |G_2 : H| \in \{2, 3\}$. Then the pair (G_1, G_2) is one of the following:*

- *If $|G_1 : H| = |G_2 : H| = 2$, then $(G_1, G_2) = (C_2, C_2)$.*
- *If $|G_1 : H| = 3$ and $|G_2 : H| = 2$, then (G_1, G_2) is one of the seven pairs of Djoković-Miller type [9].*
- *If $|G_1 : H| = |G_2 : H| = 3$, then (G_1, G_2) is one of the fifteen pairs of Goldschmidt type [12].*

Throughout this subsection, we realize the amalgamated products using the presentations given in [3] (Sections 2.1 and 2.2). In the Djoković-Miller cases, the letters h and a in the group presentations are replaced by x and y to avoid ambiguity with the use of h in this paper and to remain consistent with the presentations in the Goldschmidt cases.

Among these types, G_1^1 is distinguished by the fact that it is not two-generated.

Proposition 3.2. *The amalgamated product $\Gamma = D_6 *_{C_2} D_6$ of type G_1^1 has rank 3; thus, it is not two-generated.*

Proof. On the one hand, $\mathrm{rank}(\Gamma) \leq 3$, as seen from the presentation:

$$\Gamma = D_6 *_{C_2} D_6 = \langle c, x, y \mid c^2, x^3, y^3, (cx)^2, (cy)^2 \rangle.$$

On the other hand, Γ admits a surjective homomorphism

$$D_6 *_{C_2} D_6 \twoheadrightarrow (C_3 \times C_3) \rtimes C_2 = \langle x, y, c \mid x^3, y^3, [x, y], c^2, (cx)^2, (cy)^2 \rangle,$$

where x and y map to the generators of the two C_3 factors, and c maps to the generator of C_2. The resulting group is the generalized dihedral group of $C_3 \times C_3$, which has rank 3 by a standard result on finite groups. Therefore,

$$\mathrm{rank}(\Gamma) \geq \mathrm{rank}((C_3 \times C_3) \rtimes C_2) = 3.$$

The two inequalities confirm that $\mathrm{rank}(\Gamma) = 3$.

□

We would like to obtain two-elliptic as well as elliptic-hyperbolic generating pairs of the remaining $(3, 2)$- and $(3, 3)$-amalgamated products, with particular attention to the orders of the elliptic elements.

For two-elliptic generating pairs, we view each amalgamated product as a discrete subgroup acting on its Bass-Serre tree, either X_3 or $X_{2,3}$. If a group admits an elliptic generating pair (g, h) with $d(T_g, T_h) = 1$, then edge-transitivity implies that, up to conjugation and swapping g and h, we have $g \in G_1$ and $h \in G_2$. After adopting the convention of swapping the two factor groups in the five amalgams of G_2-type, the orders of elements in those G_1's and G_2's imply that $\mathrm{ord}(g) \in \{2, 3, 4, 6, 8\}$ and $\mathrm{ord}(h) \in \{2, 3, 4, 6, 8, 12\}$.

Not every combination of such $\mathrm{ord}(g)$ and $\mathrm{ord}(h)$ is achievable:

Proposition 3.3 ([6], Theorem C, case (3))**.** *Let g, h be elliptic tree automorphisms of prime orders p and q such that the fixed trees T_g and T_h are disjoint. Then $\Gamma = \langle g, h\rangle$ is isomorphic to the free product $C_p * C_q$. The vertex stabilizers in T_g and T_h are $\langle g\rangle \cong C_p$ and $\langle h\rangle \cong C_q$, respectively.*

When Γ is not the free product $C_3 * C_2$ of type DjM_1 or $C_3 * C_3$ of type G_1, the proposition above implies that either $\mathrm{ord}(g) \geq 4$ or $\mathrm{ord}(h) \geq 4$. This further narrows the candidate order pairs $(\mathrm{ord}(g), \mathrm{ord}(h))$ for the generators. In particular, the group $D_6 *_{C_2} V_4$ of type DjM_2^1 is not two-elliptic generated, as neither D_6 nor V_4 contains elements of order greater than 3.

For each of the remaining eighteen amalgams $\Gamma = G_1 *_H G_2$ and for each pair of orders (n, m), where $n \in \{2, 3, 4, 6, 8\}$, $m \in \{2, 3, 4, 6, 8, 12\}$, and $\max\{n, m\} \geq 4$, we are concerned whether there exist $g \in G_1$ and $h \in G_2$ such that $(\mathrm{ord}(g), \mathrm{ord}(h)) = (n, m)$ and $\langle g, h\rangle = \Gamma$. For a chosen pair (g, h), the following saturation procedure determines whether it generates Γ.

Proposition 3.4. *Suppose that $G_1 *_H G_2$ is an amalgamated product of finite groups, $g \in G_1$, $h \in G_2$, and let $K_1^{(0)} = \langle g\rangle$, $K_2^{(0)} = \langle h\rangle$. Iteratively define*

$$K_1^{(i+1)} = \langle K_1^{(i)}, K_2^{(i)} \cap H\rangle,\ K_2^{(i+1)} = \langle K_2^{(i)}, K_1^{(i)} \cap H\rangle.$$

Then $\langle g, h\rangle = G_1 *_H G_2$ if and only if there is $N < \infty$ such that $K_1^{(N)} = G_1$ and $K_2^{(N)} = G_2$.

Proof. The "if" part is clear. For the "only if" part, assume $\langle g, h\rangle = G_1 *_H G_2$. We need to show that $K_1^{(N)} = G_1$ for some N; the argument for $K_2^{(N)} = G_2$ is the same. Since G_1 is finite, it suffices to show that, for each $\gamma \in G_1$, we have $\gamma \in K_1^{(N)}$ for some N.

Since $\langle g, h\rangle = G_1 *_H G_2$, γ can be expressed as a word in g and h. By the Fundamental Theorem of Graph of Groups, we can consider $G_1 *_H G_2$ as a quotient of the free product $G_1 * G_2$, and the word

$$\gamma = g_{G_1}^{i_1} \cdot h_{G_2}^{i_2} \cdot g_{G_1}^{i_3} \cdot \dots$$

can be reduced by either (1) replacing consecutive letters $(\gamma_{1G_i}\gamma_{2G_i})$ with their product $(\gamma_1\gamma_2)_{G_i}$ for $i = 1, 2$, or (2) transferring a letter representing an element of H from one vertex group to the other.

In the initial expression of γ, each letter is in $K_1^{(0)}$ or $K_2^{(0)}$. Suppose, for example, that at some stage a letter $(\gamma_1)_{G_1} \in K_1^{(i)} \cap H$ is transferred to G_2. Under the identification of the two copies of H, the same element γ_1 may then be regarded as an element of G_2. If it is combined with a letter lying in $K_2^{(i)}$, the resulting product lies in $K_2^{(i+1)} = \left\langle K_2^{(i)},\ K_1^{(i)} \cap H \right\rangle$. The same argument applies with G_1 and G_2 interchanged.

Since the sequences

$$K_j^{(0)} \leq K_j^{(1)} \leq K_j^{(2)} \leq \cdots, \qquad j = 1, 2,$$

are increasing, it follows inductively that every letter appearing during the reduction lies in some $K_1^{(N)}$ or $K_2^{(N)}$ with finite N. The reduction terminates after finitely many steps, and since the represented element γ lies in G_1, its reduced form is the single letter $(\gamma)_{G_1}$. Hence $\gamma \in K_1^{(N_\gamma)}$ for some $N_\gamma < \infty$. The proposition is proved by taking the maximum of N_γ for $\gamma \in G_1$ or $\gamma \in G_2$.

□

Using Proposition 3.4, we test in *GAP* whether a two-elliptic generating pair exists for each candidate pair of orders.

Now we turn to the elliptic-hyperbolic generating pairs with $\ell(h) = 2$ and $d(T_g, A_h) = 1$. While a finite computer search is still possible, many of the possible values of $\mathrm{ord}(g)$ are already covered by the two-elliptic case: if (g, h) is a two-elliptic generating pair with $d(T_g, T_h) = 1$, then (g, gh) is an elliptic-hyperbolic generating pair with translation length $\ell(gh) = 2$. The two-elliptic order pairs we obtained appear to determine the answer to the elliptic-hyperbolic generating-pair existence problem for almost every value of $\mathrm{ord}(g)$ and for every amalgamated product, except in the following cases:

(1) $\mathrm{ord}(g) = 2$ and Γ of types DjM_2^1, G_2, and G_3,

(2) $\mathrm{ord}(g) = 3$ and Γ of types DjM_2^1, DjM_3, DjM_5, G_1^3, and G_2^4, and

(3) $\mathrm{ord}(g) = 6$ and Γ of type G_2^1.

Computation suggests that generating pairs in case (1) are still possible:

Proposition 3.5. *There exist elliptic-hyperbolic pairs $g, h \in \mathrm{Aut}(X_3)$ with $\mathrm{ord}(g) = 2$, $\ell(h) = 2$, and $d(T_g, A_h) = 0$ that generate the amalgamated products of types DjM_2^1, G_2, and G_3.*

Proof. **Case DjM_2^1.** Use the standard presentation:

$$D_6 *_{C_2} V_4 = \langle x, p, y \mid x^3, p^2, y^2, (xp)^2, (yp)^2 \rangle.$$

Then (p, xy) is an eligible generating pair:

Element	x	y
Recovery	$p(xy)p(xy)^{-1}$	$x^{-1}(xy)$

Case G_2. Use the standard presentation:

$$D_{12} *_{V_4} A_4 = \langle c, d, x, y \mid c^2, d^2, [c, d], x^3, y^3, (cx)^2, [d, x], (cy)^3, dy^{-1}cy \rangle.$$

Then $(g, h) = (d, xy)$ is an eligible generating pair:

Element	c	x	y
Recovery	$(xy)^{-1}d(xy)d$	$c(xy)d(xy)^{-1}$	$x^{-1}(xy)$

Case G_3. Use the standard presentation:

$$S_4 *_{D_8} S_4 = \langle c, d, x, y \mid c^2, d^4, (cd)^2, x^3, (dx^{-1})^2, cxdx, y^3, (dy^{-1})^2, cdydy \rangle.$$

Then $(g, h) = (c, yx)$ is an eligible generating pair:

Element	d	x	y
Recovery	$c(yx)^{-1}c(yx)$	$d(xy)d^2(xy)^{-1}$	$x^{-1}(xy)$

□

By contrast, the conditions in cases (2) and (3) cannot be satisfied.

Proposition 3.6. *There are no elliptic-hyperbolic pairs $g, h \in \mathrm{Aut}(X_3)$ with $\mathrm{ord}(g) = 3$ (or $\mathrm{ord}(g) = 6$), $\ell(h) = 2$, and $d(T_g, A_h) = 0$ that generate the amalgamated products of types DjM_2^1, DjM_3, DjM_5, G_1^3, and G_2^4 (or type G_2^1, respectively).*

Proof. Suppose such a pair (g, h) exists. Since $\mathrm{ord}(g) = 3$ (or 6, in the G_2^1 case), g fixes only a single vertex on A_h and permutes the three adjacent vertices. Consequently, either gh or gh^{-1} fixes a vertex adjacent to T_g. Thus, (g, gh) (or (g, gh^{-1})) is a two-elliptic generating pair of the amalgamated product, with $d(T_g, T_{gh}) = 1$ and $\mathrm{ord}(g) = 3$. However, these two-elliptic scenarios were excluded earlier by the case-by-case computation using Proposition 3.4.

□

This completes the discussion of elliptic-hyperbolic generating pairs for each possible elliptic order in each faithful amalgamated product.

3.3. Two-generator Discrete Subgroups with Small Geometric Quantities

Combining our discussion in Subsections 3.1 and 3.2, we obtain the following classification of two-generator discrete subgroups of $\mathrm{Aut}(X_3)$.

Theorem 3.3. *Let $\Gamma = \langle g, h\rangle < \mathrm{Aut}(X_3)$ be a two-generator discrete subgroup, and suppose that its action on the minimal Γ-invariant subtree is faithful.*

Two-elliptic case. *If g, h are elliptic and*

$$d(T_g, T_h) = 1,$$

then Γ is D_∞, one of the seven Djoković-Miller amalgams other than DjM^1_2, or one of the fifteen Goldschmidt amalgams other than G^1_1. Moreover,

$$\mathrm{ord}(g),\ \mathrm{ord}(h) \in \{2, 3, 4, 6, 8, 12\}.$$

Elliptic–hyperbolic case with $\ell(h) = 1$. *If g is elliptic, h is hyperbolic, and edge inversions are allowed, with*

$$d(T_g, A_h) = 0, \quad \ell(h) = 1,$$

then Γ is D_∞ or one of the seven Djoković-Miller amalgams. Moreover,

$$\mathrm{ord}(g) \in \{2, 3, 4, 6\}.$$

Elliptic–hyperbolic case with $\ell(h) = 2$. *If g is elliptic, h is hyperbolic, and*

$$d(T_g, A_h) = 0, \quad \ell(h) = 2,$$

then Γ is D_∞, one of the seven Djoković-Miller amalgams, one of the fifteen Goldschmidt amalgams other than G^1_1, or one of the faithful 2-HNN extensions classified in Theorem 3.1. Moreover,

$$\mathrm{ord}(g) \in \{2, 3, 4, 6, 8, 12\},$$

and in the HNN cases $\mathrm{ord}(g) \in \{2, 4\}$.

Conversely, every listed isomorphism type admits a generating pair satisfying the corresponding geometric conditions.

Proof. First, we identify each $\Gamma < \mathrm{Aut}(X_3)$ satisfying the conditions as a faithful amalgamated product or HNN-extension:

Two-Elliptic case. Let P denote the path of length 1 connecting T_g and T_h. Then the group Γ acts on the subforest $\bigcup_{\gamma\in\Gamma} \gamma.P \subset X_3$, which is indeed a subtree. Its quotient graph by Γ is an edge with two vertices, implying Γ is an amalgamated product. The valency restriction bounds the amalgam indices, while faithfulness of the action requires the amalgam to be faithful. Therefore, Γ is one of the faithful $(2, 2)$-, $(3, 2)$-, or $(3, 3)$-amalgamated products.

Elliptic–hyperbolic case with $\ell(h) = 2$. The group Γ acts on the subforest $\bigcup_{\gamma\in\Gamma} \gamma.A_h \subset X_3$. Since $d(T_g, A_h) = 0$, it is again a subtree. Its quotient graph by Γ is the circuit $A_h/\langle h\rangle$ of length $\ell(h) = 2$, up to further edge identifications. Therefore, it is either a circuit of length at most 2, or an edge with two vertices.

In the edge case, Γ is one of the faithful $(2, 2)$-, $(3, 2)$-, or $(3, 3)$-amalgamated products discussed above. In the circuit case, each vertex group should contain the two edge groups with indices $(1, 1)$ or $(1, 2)$, as required by the valency of the covering tree. Since the indices must also be compatible with the circuit, the circuit length should be 2, and the index pair for each vertex should be $(1, 2)$. After reducing the edge with index 1 and merging the two vertices, the result is a loop with an edge group of index 2, corresponding to a faithful 2-HNN extension.

Elliptic–hyperbolic case with $\ell(h) = 1$. If $\ell(h) = 1$ instead, then the group Γ contains edge inversions. After barycentric subdivision of the edge, the quotient graph becomes a "half-edge", and the index $|G_2 : H| = 2$ for the barycenter stabilizer G_2. This narrows the possibilities to faithful $(2, 2)$- or $(3, 2)$-amalgamated products.

We now identify the groups appearing in each case. The faithful amalgams are classified in Theorem 3.2 as D_∞, the Djoković-Miller amalgams, and the Goldschmidt amalgams. The faithful HNN extensions are classified in Theorem 3.1

as an explicit infinite family. By Proposition 3.2, the Goldschmidt amalgam G_1^1 does not occur in any of the three cases. By the remark after Proposition 3.3, the Djoković-Miller amalgam DjM_2^1 does not occur in the two-elliptic case.

All remaining amalgams admit two-elliptic generating pairs with the required orders, as verified by the computation using Proposition 3.4. Proposition 3.5, together with the preceding remark, provides elliptic-hyperbolic generating pairs with translation length 2 and the required orders. For Djoković-Miller amalgams and elliptic-hyperbolic generating pairs with translation length 1, the elliptic generator must come from the vertex stabilizer containing the edge stabilizer with index 3. Thus, the computation using Proposition 3.4 restricts the possible orders to $2, 3, 4, 6$. Finally, Proposition 3.1 provides elliptic-hyperbolic generating pairs with translation length 2 and the required orders in $2, 4$.

□

We can also consider the classification with the faithfulness condition replaced by the lattice condition. In fact, requiring $\Gamma < \mathrm{Aut}(X_3)$ to be a lattice is stronger:

Lemma 3.4. *Let X be a locally finite tree without leaves, and let $\Gamma < \mathrm{Aut}(X)$ be finitely generated and discrete. Then Γ is a lattice if and only if its minimal Γ-invariant subtree satisfies $T_\Gamma = X$.*

Proof. If $T_\Gamma = X$, Definition 2.6 implies that X/Γ is a finite graph; hence Γ is a lattice (indeed, a uniform lattice).

Conversely, if $T_\Gamma \subsetneq X$, a component of $X \setminus T_\Gamma$ contains vertices at every distance from T_Γ since X has no leaves, and the distance is Γ-invariant. Fix such a component. The stabilizer of each of its vertices is a subgroup of the stabilizer of its root vertex $v \in \mathcal{V}(T_\Gamma)$. Consequently, the covolume diverges, so Γ is not a lattice.

□

As a consequence of Lemma 3.4, if Γ is a lattice in $\mathrm{Aut}(X_3)$, then Γ acts on its minimal tree (which is X_3 itself) faithfully. Theorem 3.3 then implies the following classification of $\mathrm{Aut}(X_3)$-lattices.

Corollary 3.1. *Let $\Gamma = \langle g, h\rangle < \mathrm{Aut}(X_3)$ be a two-generator lattice.*

Two-elliptic case. *If g, h are elliptic and*

$$d(T_g, T_h) = 1,$$

then Γ is one of the fifteen Goldschmidt amalgams other than G_1^1. Moreover,

$$\mathrm{ord}(g),\ \mathrm{ord}(h) \in \{2, 3, 4, 6, 8, 12\}.$$

Elliptic–hyperbolic case with $\ell(h) = 1$. *If g is elliptic, h is hyperbolic, and edge inversions are allowed, with*

$$d(T_g, A_h) = 0, \quad \ell(h) = 1,$$

then Γ is one of the seven Djoković-Miller amalgams. Moreover,

$$\mathrm{ord}(g) \in \{2, 3, 4, 6\}.$$

Elliptic–hyperbolic case with $\ell(h) = 2$. *If g is elliptic, h is hyperbolic, and*

$$d(T_g, A_h) = 0, \quad \ell(h) = 2,$$

then Γ is one of the fifteen Goldschmidt amalgams other than G_1^1 or one of the faithful 2-HNN extensions classified in Theorem 3.1. Moreover,

$$\mathrm{ord}(g) \in \{2, 3, 4, 6, 8, 12\},$$

and in the HNN cases $\mathrm{ord}(g) \in \{2, 4\}$.

Conversely, every listed isomorphism type admits a generating pair satisfying the corresponding geometric conditions.

Proof. By Lemma 3.4, Γ acts faithfully on its minimal subtree, and Theorem 3.3 determines the possibilities for Γ. Among these amalgams and HNN extensions, the minimal subtree of D_∞ is a bi-infinite path, which does not meet the requirement. The minimal subtrees of the Goldschmidt amalgams are all X_3, and these are compatible with the two-elliptic case and the elliptic-hyperbolic case with translation length 2. Since the HNN extensions are obtained by shrinking an edge from a circuit of length 2 with indices $(1, 2)$, their minimal subtrees are also X_3. Finally, the minimal subtrees of the Djoković-Miller amalgams are $X_{2,3}$, which is the barycentric subdivision of X_3 and is compatible with the elliptic-hyperbolic case with translation length 1.

□

As a final remark, let us discuss the relationship of Theorem 3.3, especially the elliptic-hyperbolic cases, to two-hyperbolic generating pairs. In [10, Theorem 3.3], we showed that a hyperbolic generating pair (g, h) with sufficiently long axis intersection can be transformed, by Nielsen moves, into an elliptic–hyperbolic pair, and explicitly determined the geometric quantities of the resulting pair. In particular, when $\gcd(\ell(g), \ell(h)) \leq 2$ and $\ell(A_g \cap A_h) \geq \ell(g) + \ell(h) - \gcd(\ell(g), \ell(h))$, the reduction result produces the small-geometry cases treated in Theorem 3.3.

4. Geometry of Quotient Graphs by Two Generators

We now turn to general two-generator discrete subgroups of tree automorphisms. This section focuses on a preliminary piece of rigidity: for an elliptic-hyperbolic generating pair (g, h) with fixed $\ell(h)$ and $d(T_g, A_h)$, we show that the minimal quotient graph for $\Gamma = \langle g, h\rangle$ is constrained.

Theorem 4.1. *Let X be a tree, $g, h \in \mathrm{Aut}(X)$, and let $\Gamma = \langle g, h\rangle$ be a discrete and faithful subgroup. Suppose g is elliptic with fixed tree T_g, and h is hyperbolic with translation length $l = \ell(h)$, axis A_h, and distance $d = d(T_g, A_h)$. Then the quotient graph Y of the minimal Γ-subtree by Γ has cyclomatic number $\mathrm{b}_1(Y) \leq 1$; that is, Y is either a tree or unicyclic.*

In addition, Y must be obtained by identifying edges of the (l, d)-tadpole graph Y' (a circuit of length l with an attached path of length d). Furthermore, if Y is unicyclic, then the circuit in Y must be homotopic to the image of the circuit in the tadpole graph Y' under the edge identifications.

Proof. For the first claim, we consider the fundamental group homomorphism

$$\varphi : \Gamma = \pi_1(Y, \mathcal{G}) \to \pi_1(Y) = F_{\mathrm{b}_1(Y)},$$

obtained by forgetting the finite vertex groups of the quotient graph of groups $(Y, \mathcal{G})$.

On the one hand, this forgetful homomorphism is surjective. On the other hand, $\Gamma = \langle g, h\rangle$, with g elliptic. Thus $\varphi(g) = 1$, and $\varphi(\Gamma)$ is either trivial or an infinite cyclic group $\langle\varphi(h)\rangle$. In particular, the rank of the free group $\pi_1(Y)$ is at most one, so $\mathrm{b}_1(Y) \leq 1$.

For the assertion about the quotient graph, let P be the path joining T_g and A_h. Then the group Γ acts on the subtree

$$X_0 = \bigcup_{\gamma\in\Gamma} (\gamma.A_h \cup \gamma.P).$$

Its quotient graph, X_0/Γ, is obtained by identifying edges in the vertex sum of $A_h/\langle h\rangle$, a circuit of length $\ell(h)$, with P, a path of length $d(T_g, A_h)$. Thus it is obtained from the tadpole graph Y' by edge identifications.

In this construction, h corresponds to the circuit of length l in Y', and after further edge identifications and forgetting the vertex groups, $\varphi(h) \in \pi_1(Y)$ corresponds to the image of that circuit in Y'. Therefore, if Y is unicyclic, then its circuit must be the image of the circuit in the tadpole graph Y' under the edge identifications.

□

Theorem 4.1 implies that Y is indeed obtained by *folding* the edges of the (l, d)-tadpole graph, i.e., repeatedly identifying adjacent edges, as well as their endpoints that are not incident with the other edge (if any). The theorem further gives an upper bound on the size of the quotient graph:

Corollary 4.1. *If* $\mathrm{b}_1(Y) = 0$*, then* $|\mathcal{E}(Y)| \leq d(T_g, A_h) + \frac{1}{2}\ell(h)$.
If $\mathrm{b}_1(Y) = 1$*, then the girth of* Y *is at most* $\ell(h)$*, and*

$$|\mathcal{E}(Y)| \leq d(T_g, A_h) + \frac{1}{2}(\ell(h) + \mathrm{girth}(Y)).$$

When $d(T_g, A_h) = 0$*, this bound is sufficient.*

Proof. If $\mathrm{b}_1(Y) = 0$, then the circuit collapses to a tree, so each edge coming from the circuit arises from identifying at least two edges. If $\mathrm{b}_1(Y) = 1$, then the length of the remaining circuit is at most $\ell(h)$, and each edge not lying on the circuit also arises from identifying at least two edges. This gives the required upper bound on the size of the graph. □

5. Constructions of Two-generator Graphs of Groups

Despite the restriction on quotient graphs given by Theorem 4.1, the vertex stabilizers of general two-generator discrete subgroups $\langle g, h\rangle < \mathrm{Aut}(X)$—even under the faithfulness condition—can be complicated:

- If $\ell(h) = 4$, $d(T_g, A_h) = 0$, and X is trivalent, then $\Gamma = \langle g, h\rangle$ could be an amalgam of three finite groups:

  $$\Gamma = G_0 *_{H_1} G_1 *_{H_2} G_2,$$

  where $\{|G_1 : H_1|, |G_2 : H_2|\} = \{1, 2\}$, and $|G_0 : H_1|, |G_2 : H_2| \leq 3$.
- If inversions are allowed and either $\ell(h) = 3$ with $d(T_g, A_h) = 0$ or $\ell(h) = 1$ with $d(T_g, A_h) = 1$, we obtain amalgams of the same form as in the previous case.
- If $\ell(h) = 2$, $d(T_g, A_h) = 0$, and X is four-regular or $(3, 4)$-biregular, then there is a family of two-generator faithful $(4, 3)$-amalgamated products, which in turn contains a family of three-group amalgams $G_0 *_{H_1} G_1 *_{H_2} G_2$ equivalent to the above.

This family of amalgamated products is infinite and appears to be impractical to fully classify. Instead, we aim to construct explicit families of two-generator graphs of groups satisfying the following index conditions:

Definition 5.1. *Let* Y *be a graph obtained by folding an* (l, d)*-tadpole. For each edge* e *and incidence* $v \in \{\alpha_e, \omega_e\}$*, assign*

$$n_{e,v} \in \{1, 2, 3\}.$$

We call $\{n_{e,v}\}$ *an* admissible index assignment *if it satisfies the following conditions:*

- $n_{e,v} = 3$ *if* $\mathrm{val}(v) = 1$*, and* $n_{e,v} \in \{1, 2\}$ *if* $\mathrm{val}(v) \geq 2$.
- *If* Y *is a path or a circuit, then at least one* $n_{e,v} = 2$.
- *If* Y *is unicyclic and contains a circuit* C*, let* $v^+(e)$ *and* $v^-(e)$ *be the left and right endpoints of* $e \in \mathcal{E}(C)$*. Then*

  $$\prod_{e\in\mathcal{E}(C)} n_{e,v^+(e)} = \prod_{e\in\mathcal{E}(C)} n_{e,v^-(e)}.$$

- *The* folding condition*: if* e *is the result of folding at least two edges incident with the same vertex* v *in the* (l, d)*-tadpole, then* $n_{e,v} > 1$.

Later in Section 6, we will use these constructions to show the richness of two-generator discrete $\mathrm{Aut}(X)$-subgroups. Before proceeding to the construction, it is worth mentioning that the edge groups are often constructed as elementary abelian 2-groups:

$$H = \langle t_1, \ldots, t_k \mid t_i^2 = (t_i t_j)^2 = 1\rangle \cong (C_2)^k.$$

We identify H with the vector space $V_H = \mathbb{F}_2^k$, which has dual space $V_H^* = \mathrm{Hom}(V_H, \mathbb{F}_2)$. For any $f \in V_H^*$, we denote its kernel by $\ker(f) = \{h \in H \mid f(h) = 0\}$.

The following lemma provides conditions ensuring that H is core-free in a larger group. In our constructions, this is usually sufficient to demonstrate the faithfulness of the Γ-action on its Bass-Serre tree, thereby ensuring that Γ is a subgroup of $\mathrm{Aut}(X)$.

Lemma 5.1. *Suppose $H < \Gamma$ for some ambient group Γ. Assume the following:*

- *There exist elements $r_1, \dots, r_l \in \Gamma$ such that $(\bigcap r_i H r_i^{-1}) \cap H = \ker(t_1^*, \dots, t_n^*)$ for some non-zero $t_1^*, \dots, t_n^* \in V_H^*$.*
- *There exist elements $x_1, \dots, x_j \in \Gamma$, and for each $i = 1, \dots, j$, there exists $\varphi_i \in \mathrm{GL}(k, 2)$ such that $x_i t x_i^{-1} = \varphi_i(t)$ for all $t \in H \cap x_i^{-1} H x_i$.*

Let $\mathcal{W} \subset \mathrm{GL}(k, 2)$ be the set of automorphisms induced by words in the generators $\{x_i\}$. If the set of pulled-back functionals $\{W^T t_i^* \mid W \in \mathcal{W},\ i = 1, \dots, n\}$ spans V_H^*, then H is core-free *in Γ, i.e.,*

$$\bigcap_{\gamma \in \Gamma} \gamma H \gamma^{-1} = \{1\}.$$

Proof. Let $K = \bigcap_{\gamma \in \Gamma} \gamma H \gamma^{-1}$ be the core of H in Γ. Let $W \in \mathcal{W}$ be the linear transformation corresponding to the conjugation action of w^{-1}, where

$$w = x_{i_m}^{\pm} \dots x_{i_1}^{\pm}.$$

Denote by w_i the partial product $x_{i_k}^{\pm} \dots x_{i_1}^{\pm}$. For any t in

$$\left(\bigcap_{i=1}^{l} (w r_i) H (w r_i)^{-1} \right) \cap \left(\bigcap_{k=0}^{m} w_k H w_k^{-1} \right),$$

each intermediate conjugate $w_k^{-1} t w_k$ lies in H, so the successive conjugation relations $\mathrm{ad}_{x_{i_{k+1}}} = \varphi_{i_{k+1}}$ apply. Hence $w^{-1} t w = W(t)$ as an element of V_H. Moreover,

$$w^{-1} t w \in H \cap \left(\bigcap_{i=1}^{l} r_i H r_i^{-1} \right),$$

so $W(t) \in \ker(t_1^*, \dots, t_n^*)$, meaning that

$$\langle t_i^*, W(t) \rangle = 0 \iff \langle W^T t_i^*, t \rangle = 0,\ i = 1, \dots, n.$$

Thus, $t \in \ker(W^T t_1^*, \dots, W^T t_n^*)$ for every $W \in \mathcal{W}$. Since the collection $\{W^T t_i^* \mid W \in \mathcal{W},\ i = 1, \dots, n\}$ spans V_H^*, an element t lies in the intersection of these kernels if and only if $t = 1$ (the identity in H). Specifically,

$$K \subseteq \bigcap_{W \in \mathcal{W}, 1 \le i \le n} \ker(W^T t_i^*) = \{1\}.$$

We conclude that the core of H is trivial, and thus H is core-free in Γ.

□

Another lemma is required to recover a generating pair from a graph of groups with the desired geometric quantities $\ell(h)$ and $d(T_g, A_h)$:

Lemma 5.2. *We say that a collection of subtrees $\{T_i\}$ is* bridge-reduced *if (i) each T_i contains exactly one vertex v_i minimizing the distance to $\bigcup_{j \neq i} T_j$, and (ii) each v_i is a leaf of the subtree induced by them.*

(1) If $h_1, \dots, h_m \in \mathrm{Aut}(X)$ are elliptic automorphisms and $\{T_{h_1}, \dots, T_{h_m}\}$ is bridge-reduced, then their product

$$h = h_1 \dots h_m$$

is hyperbolic, and

$$\ell(h) = \sum_{i=1}^{m} d(T_{h_i}, T_{h_{i+1}}),\ h_{m+1} = h_1.$$

(2) In addition to (1), if T^- and T^+ are subtrees of X such that $\{T^-, T^+, T_{h_1}, \dots, T_{h_m}\}$ is bridge-reduced, then

$$d(T^-, h.T^+) = d(T^-, T_{h_1}) + \sum_{i=1}^{m-1} d(T_{h_i}, T_{h_{i+1}}) + d(T_{h_m}, T^+).$$

(3) In addition to (2), if $T^- = v^-$ and $T^+ = v^+$ are vertices on the axis of a hyperbolic automorphism $h_0 \in \mathrm{Aut}(X)$, with $d(v^-, v^+) = \ell(h_0) + 1$, then

$$h' = h_0 h_1 \dots h_m$$

is hyperbolic with translation length

$$\ell(h') = d(v^-, h.v^+) - 1.$$

The proof is a straightforward adaptation of Exercise 2.6 in [2].

5.1. Two-generator Subgroups with Elementary Quotient Graphs

We will first describe the construction of two-generator subgroups of $\mathrm{Aut}(X)$ with *elementary* quotient graphs, meaning a path, a circuit, or a star graph. These examples contain the main ideas needed for the general construction of two-generator $\mathrm{Aut}(X)$-subgroups.

5.1.1. Paths of Length ≥ 2

By Theorem 3.2, there are only finitely many faithful amalgamated products of indices ≤ 3, so there are only finitely many paths of groups of length 1. For this reason, the infinite family we will construct will apply to paths of length at least 2.

Proposition 5.1. *For any $n \geq 2$, any $j = 1, \dots, n$, and every admissible assignment of indices $n_{e,v}$ on a path Y of length n, there exists an infinite family of two-generator subgroups $\Gamma = \langle g, h \rangle < \mathrm{Aut}(X_4)$ with arbitrarily large vertex stabilizers and the following properties:*

- *The generator g is elliptic of order* $\mathrm{ord}(g) \in \{3, 6\}$.
- *The generator h is hyperbolic, with $\ell(h) = 2j$ and $d(T_g, A_h) = n - j$.*
- *The minimal quotient graph for Γ is the path Y, and the associated graph of groups satisfies $[\Gamma_v : \Gamma_e] = n_{e,v}$ for each edge e incident with a vertex v. In particular, this includes the case when Y is a reduced quotient.*

The construction. Let the vertices of Y be $v_0, v_1, \dots, v_n$, and the edges be $e_0, e_1, \dots, e_{n-1}$. We aim to construct a family of vertex groups $G_i = \mathcal{G}(v_i)$ and edge groups $H_i = \mathcal{G}(e_i)$, so that the fundamental group of the graph of groups $(Y, \mathcal{G})$,

$$\Gamma = \pi_1(Y, \mathcal{G}) = G_0 *_{H_0} G_1 *_{H_1} \cdots *_{H_{n-1}} G_n$$

satisfies the requirements in the proposition.

We first define the vertex groups G_k for $k = 1, \dots, n-1$. Let

$$m_k = \#\{1 \leq i \leq k \mid n_{e_{i-1}, v_i} = 2\},\ m'_k = \#\{1 \leq i \leq k-1 \mid n_{e_i, v_i} = 2\},$$

so

$$m_{k+1} - m_k,\ m'_{k+1} - m'_k \in \{0, 1\},$$

depending on the values $n_{e_k, v_{k+1}}$ and n_{e_k, v_k}. For N large enough, let

$$G_k = \langle t_1, \dots, t_{2N-1}, r_{m'_k+1}, \dots, r_{m'_n}, s_1, \dots, s_{m_k} \rangle,$$

which is abelian and generated by involutions. For $k = 0, \dots, n-1$, let

$$H_k = \langle t_1, \dots, t_{2N-1}, r_{m'_{k+1}+1}, \dots, r_{m'_n}, s_1, \dots, s_{m_k} \rangle,$$

so $H_k < G_k$ and $H_k < G_{k+1}$, with $|G_k : H_k| = n_{e_k,v_k}$ and $|G_{k+1} : H_k| = n_{e_k,v_{k+1}}$. In particular,

$$H_0 = \langle t_1, \dots, t_{2N-1}, r_1, \dots, r_{m'_n} \rangle,$$

and

$$H_{n-1} = \langle t_1, \dots, t_{2N-1}, s_1, \dots, s_{m_{n-1}} \rangle.$$

Define $G_0 = H_0 \rtimes \langle x \rangle$ and $G_n = H_{n-1} \rtimes \langle y \rangle$, with $\mathrm{ord}(x) = \mathrm{ord}(y) = 3$, and the conjugation relations on the generators are as follows:

t	t_{2i-1}	t_{2i}	r_i	$(i = 1, \dots, m'_n)$
t^x	t_{2i}	r_i	t_{2i-1}	
t	t_{2N-1}	t_{2i-1}	t_{2i}	$(i = m'_n + 1, \dots, N-1)$
t^x	t_{2N-1}	t_{2i}	$t_{2i-1}t_{2i}$	

t	t_1	t_{2i}	t_{2i+1}	$(i = 1, \dots, N - m_{n-1} - 1)$
t^y	t_1	t_{2i+1}	$t_{2i}t_{2i+1}$	
t	t_{2i}	t_{2i+1}	s_{N-i}	$(i = N - m_{n-1}, \dots, N-1)$
t^y	t_{2i+1}	s_{N-i}	t_{2i}	

From the construction, one sees that Γ has an admissible index assignment; the covering tree has valency at most 4, and Γ acts on X_4. We will show that Γ is two-generated, has the desired geometric quantities, and acts faithfully on its covering tree.

Proof. **Generating Pairs.** For $j = n$, let $(g, h) = (x, xyt_1)$. Both x and yt_1 are elliptic, and the graph of groups suggests

$$d(T_x, T_{yt_1}) = n.$$

Lemma 5.2 implies that h is hyperbolic with $\ell(h) = 2n$. In addition, g is elliptic of order 3, and $d(T_g, A_h) = 0$. The original generators y, t_i, r_i and s_i are recovered as follows:

Generator	t_1	y	t_{2i}	t_{2i+1}	r_i	s_i
Recovery	$(g^{-1}h)^3$	$(g^{-1}h)^4$	$xt_{2i-1}x^{-1}$	$yt_{2i}y^{-1}$	$xt_{2i}x^{-1}$	$yt_{2(N-i)+1}y^{-1}$

For $j < n$, let $(g, h) = (yt_1, xs_{m_j})$. Similarly, we have

$$d(T_{xs_{m_j}}, T_{yt_1}) = j,$$

and h is hyperbolic with $\ell(h) = 2j$ by Lemma 5.2; g is elliptic of order 6, and $d(T_g, A_h) = n - j$. The original generators x, y, t_i, r_i and s_i are recovered as follows:

Generator	t_1	y	t_{2i}	t_{2i+1}	s_i	x	r_i
Recovery	g^3	g^4	$ht_{2i-1}h^{-1}$	$yt_{2i}y^{-1}$	$yt_{2(N-i)-1}y^{-1}$	hs_{m_j}	$xt_{2i}x^{-1}$

Action Faithfulness. It suffices to show that H_0 is core-free in Γ:

$$\bigcap_{\gamma \in \Gamma} \gamma H_0 \gamma^{-1} = \{1\}.$$

Indeed, the conjugation relations give

$$H_0 \cap yH_0y^{-1} \cap y^{-1}H_0y = \ker(r_1^*, \dots, r_{m'_n}^*, t_{2(N-m_{n-1})}^*, \dots, t_{2N-1}^*).$$

As long as $\max(m'_n, m_{n-1}) > 0$, this kernel is non-trivial. Alternately applying the pullbacks induced by x- and y-conjugation to the covectors defining this kernel gives all generators of $V_{H_0}^*$, namely

$$V_{H_0}^* = \mathrm{span}(t_1^*, \dots, t_{2N-1}^*, r_1^*, \dots, r_{m'_n}^*).$$

By Lemma 5.1, H_0 is core-free, and the Γ-action on the Bass-Serre tree is faithful.

□

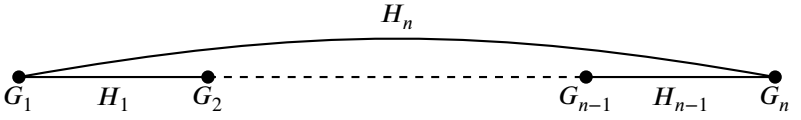
5.1.2. Circuits

The infinite family of faithful 2-HNN extensions in Theorem 3.1 has loop quotient graphs. Modifying the construction yields a family of two-generator subgroups whose quotient graphs are circuits of any length.

Proposition 5.2. *For any $n \geq 1$ and every admissible assignment of indices $n_{e,v}$ on a circuit Y of length n, there exists an infinite family of two-generator subgroups $\Gamma = \langle g, h \rangle < \mathrm{Aut}(X_4)$ with arbitrarily large vertex stabilizers and the following properties:*

- *The generator g is elliptic of order $\mathrm{ord}(g) = 2$.*
- *The generator h is hyperbolic, with $\ell(h) = n$ and $d(T_g, A_h) = 0$.*
- *The minimal quotient graph for Γ is the circuit Y, and the associated graph of groups satisfies $[\Gamma_v : \Gamma_e] = n_{e,v}$ for each edge e incident with a vertex v.*

The construction. Let the vertices of Y be $v_1, v_2, \ldots, v_n$, and the edges be $e_1, e_2, \ldots, e_n$. We aim to construct a family of vertex groups $G_i = \mathcal{G}(v_i)$ and edge groups $H_i = \mathcal{G}(e_i)$, such that the fundamental group Γ of the graph of groups $(Y, \mathcal{G})$ satisfies the requirements in the proposition:



Let us define the vertex and edge groups: without loss of generality, let v_1 be the vertex with the smallest vertex group order, as implied by the numbers $n_{e,v}$. Let

$$m_k = \#\{1 \leq i \leq k-1 \mid n_{e_i, v_{i+1}} = 2\},\ m'_k = \#\{1 \leq i \leq k-1 \mid n_{e_i, v_i} = 2\},$$

so

$$m_{k+1} - m_k,\ m'_{k+1} - m'_k \in \{0, 1\},\ m_{n+1} = m'_{n+1} := m.$$

For $N \geq 3$, we define

$$G_k = \langle t_{m'_k+1}, \ldots, t_{mN+m_k+1} \mid t_i^2 = 1,\ (t_i t_{i+mN})^2 = t_{i+m+1},\ (t_i t_{i+j})^2 = 1, \forall j \neq mN \rangle,$$

and

$$H_k = \langle t_{m'_{k+1}+1}, \ldots, t_{mN+m_k+1} \rangle,$$

with the same relators. Note that the choice of v_1 implies that $m_k \geq m'_k$, so every G_k contains non-trivial commutator relations. On the other hand, $m_k - m'_k \leq m$, so each t_{i+m+1} on the right-hand side of a commutator relation does not appear in the left-hand side of another commutator $(t_j t_{j+mN})^2$. This guarantees the groups G_k do not collapse.

Note that the subgroup relations $H_k < G_k$ and $H_k < G_{k+1}$ hold, with $|G_k : H_k| = n_{e_k, v_k}$ and $|G_{k+1} : H_k| = n_{e_k, v_{k+1}}$. In particular,

$$G_1 = \langle t_1, \ldots, t_{mN+1} \rangle,$$

and

$$H_n = \langle t_{m+1}, \ldots, t_{m(N+1)+\epsilon} \rangle,$$

where $\epsilon \in \{0, 1\}$, depending on whether $n_{e_n, v_1} = 1$ or 2. Define the stable letter x and the boundary injection $\varphi : H_n \to G_1$ by

$$\varphi(t_i) = x t_i x^{-1} = t_{i-m}.$$

From the construction, one sees that the index assignment on Γ is admissible, and Γ acts on X_4. We will show that Γ is two-generated, has the desired geometric quantities, and acts faithfully.

Proof. **Generating Pairs.** Let $(g, h) = (t_1, x)$; then g is elliptic of order 2, h is hyperbolic with $\ell(h) = n$, and $d(T_g, A_h) = 0$. The original generators t_i are recovered as follows:

Generator	t_{i+1}	t_{i+m}
Recovery	$\varphi((t_i\varphi^{-N}(t_i))^2)$	$\varphi^{-1}(t_i)$

Action Faithfulness. It is clear from the definition of φ that

$$\bigcap_{i\in\mathbb{Z}} \varphi^i(H_1) = \bigcap_{i\in\mathbb{Z}} x^i H_1 x^{-1} = \{1\}.$$

Therefore, H_1 is core-free, and the Γ-action on the Bass-Serre tree is faithful.

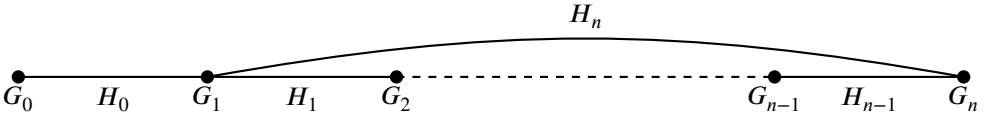
□

When Y is a unicyclic graph with leaves, we can construct two-generator subgroups with minimal quotient graph Y and abelian vertex stabilizers on the circuit. As the construction below demonstrates, this simplifies the general construction.

Corollary 5.1. *For any $n \geq 1$, $j \in \{0, 1\}$, and every admissible assignment of indices $n_{e,v}$ on an $(n, 1)$-tadpole graph (a leaf vertex attached to an n-circuit), there exists an infinite family of two-generator subgroups $\Gamma = \langle g, h\rangle < \mathrm{Aut}(X_6)$ with arbitrarily large vertex stabilizers and the following properties:*

- *The generator g is elliptic of order* $\mathrm{ord}(g) = 6$.
- *The generator h is hyperbolic, with $\ell(h) = n + 2j$ and $d(T_g, A_h) = 1 - j$.*
- *The minimal quotient graph for Γ is the tadpole graph Y, and the associated graph of groups satisfies $[\Gamma_v : \Gamma_e] = n_{e,v}$ for each edge e incident with a vertex v.*

The construction. Let the vertices of Y be $v_0, v_1, \ldots, v_n$ and the edges be $e_0, e_1, \ldots, e_n$, corresponding to vertex groups G_i and edge groups H_i. Here, the vertices v_1 through v_n form a loop, and v_0 is a leaf attached to the loop at v_1:



Again, let

$$m_k = \#\{1 \leq i \leq k-1 \mid n_{e_i,v_{i+1}} = 2\},\ m'_k = \#\{1 \leq i \leq k-1 \mid n_{e_i,v_i} = 2\},$$

so $m_{n+1} = m'_{n+1} := m$. For $i = 1, \ldots, n$, define the vertex and edge groups G_i and H_i with the same generators as in Proposition 5.2, but without the non-trivial commutators:

$$G_k = \langle t_{m'_k+1}, \ldots, t_{mN+m_k+1} \mid t_i^2 = 1, (t_i t_j)^2 = 1\rangle,$$

and

$$H_k = \langle t_{m'_{k+1}+1}, \ldots, t_{mN+m_k+1} \mid t_i^2 = 1, (t_i t_j)^2 = 1\rangle,$$

where N is any sufficiently large number. Define the boundary injection $\varphi : H_n \to G_1$, $\varphi(t_i) = x t_i x^{-1} = t_{i-m}$ as before. In particular, the group for the junction vertex v_1 (where the tail meets the circuit) is

$$G_1 = \langle t_1, \ldots, t_{mN+1}\rangle.$$

Depending on the number n_{e_0,v_1}, let $H_0 = G_1$ or an index-2 subgroup that does not contain any generator t_i with $i > 2m$. Define $G_0 = H_0 \rtimes \langle y\rangle$ with $\mathrm{ord}(y) = 3$, and the non-trivial conjugation relations are given as follows:

t	t_{2i}	t_{2i+1}	$(1 \le i \le (m-1)/2)$
t^y	t_{2i+1}	$t_{2i}t_{2i+1}$	
t	t_{m+2i-1}	t_{m+2i}	$(1 \le i \le m/2)$
t^y	t_{m+2i}	$t_{m+2i-1}t_{m+2i}$	

We will show that Γ is two-generated, has the desired geometric quantities, and acts faithfully.

Proof. **Generating Pairs.** For $j = 0$, let $(g, h) = (yt_1, x)$; then g is elliptic of order 6, h is hyperbolic with $\ell(h) = n$ and $d(T_g, A_h) = 1$. The original generators y and t_i are recovered as follows:

Generator	y	t_1	t_{2i}	t_{2i+1}	t_{i+m}
Recovery	g^4	g^3	$xyx^{-1}t_{2i-1}xy^{-1}x^{-1}$	$yt_{2i}y^{-1}$	$x^{-1}t_ix$

For $j = 1$, let $(g, h) = (yt_1, xyt_1)$; then g is elliptic of order 6, and by Lemma 5.2, h is hyperbolic with $\ell(h) = n+2$ and $d(T_g, A_h) = 0$. Note that the two generating pairs are equivalent.

Action Faithfulness. As before, the subgroup H_1 is core-free in Γ, since

$$\bigcap_{i\in\mathbb{Z}} \varphi^i(H_1) = \bigcap_{i\in\mathbb{Z}} x^i H_1 x^{-1} = \{1\}.$$

□

5.1.3. Star Graphs

To illustrate the construction of two-generator subgroups whose minimal quotient graph has at least three leaves, we discuss the simplest case, namely a star graph.

Proposition 5.3. *For any $n \ge 3$, each $j = 1, \dots, n$, and every admissible assignment of indices $n_{e,v}$ on a star graph Y of n edges, there exists an infinite family of two-generator subgroups $\Gamma = \langle g, h\rangle < \mathrm{Aut}(X_{2n})$ with arbitrarily large vertex stabilizers and the following properties:*

- *The generator g is elliptic of order* $\mathrm{ord}(g) = 6$.
- *The generator h is hyperbolic, with $\ell(h) = 2j$, and $d(T_g, A_h) = 2(n-j)-1$ when $j < n$, or $d(T_g, A_h) = 0$ when $j = n$.*
- *The minimal quotient graph for Γ is the star graph Y, and the associated graph of groups satisfies $[\Gamma_v : \Gamma_e] = n_{e,v}$ for each edge e incident with a vertex v.*

The construction. Let the vertices of Y be $v_\star$, v_0,…, v_{n-1}, and the edges be e_0,…, e_{n-1}. Here $v_\star$ is the center vertex, and e_i is incident with $v_\star$ and v_i. We aim to construct a family of vertex groups $G_\star$ and G_i, and edge groups H_i, $i = 0, \dots, n-1$, such that the fundamental group Γ of the graph of groups $(Y, \mathcal{G})$ satisfies the requirements in the proposition.

For sufficiently large N, define

$$G_\star = \langle t_0, t_1, \dots, t_{3N-2}\rangle \cong (C_2)^{3N-1}.$$

For $i = 0, \dots, n-1$, define the edge subgroups: if $n_{e_i,v_\star} = 1$, set $H_i = G_\star$. Otherwise,

$$H_0 = \langle t_i \mid i \ne 3N-2\rangle,\ H_{n-1} = \langle t_i \mid i \ne 0\rangle,$$

and

$$H_j = \langle t_i \mid i \ne 3j+2\rangle,\ j = 1, \dots, n-2.$$

Define $G_0 = H_0 \rtimes \langle x\rangle$ and $G_{n-1} = H_{n-1} \rtimes \langle y\rangle$, with $\mathrm{ord}(x) = \mathrm{ord}(y) = 3$, and the conjugation relations on the generators as follows (trivial relations omitted):

t	t_{3i}	t_{3i+1}	t_{3i+2}	$(i = 0, \dots, N-1)$
t^x	t_{3i+1}	t_{3i+2}	t_{3i}	

t	t_{3i-1}	t_{3i}	t_{3i+1}	$(i = 1, \dots, N)$
t^y	t_{3i}	t_{3i+1}	t_{3i-1}	

For $j = 1, \dots, n-2$, define $G_j = \langle z_j \rangle \rtimes H_j$, $\mathrm{ord}(z_j) = 3$, with the only non-trivial conjugation relation being

$$t_{3j+1} z_j t_{3j+1} = z_j^2.$$

The group inclusions $H_i < G_i$ and $H_i < G_\star$ define an n-star graph of groups, and the index assignment is admissible. We will show that its fundamental group Γ is two-generated, has the desired geometric quantities, and acts faithfully.

Proof. **Generating Pairs.** For $j = n$, let $(g, h) = (yt_1, yz_1 \dots z_{n-2}x)$. By Lemma 5.2:

$$\ell(h) = d(v_0, v_1) + \cdots + d(v_{n-2}, v_{n-1}) + d(v_{n-1}, v_0) = 2n,$$

and $d(T_g, A_h) = 0$. Denote $x_i := z_i \dots z_{n-2}x$; then the original generators are recovered as follows:

Generator	y	t_1	x_1	t_{3i-1}	t_{3i}	t_{3i+1}	z_i	x_{i+1}
Recovery	g^4	g^3	$y^{-1}h$	$x_i t_{3i-2} x_i^{-1}$	$y t_{3i-1} y^{-1}$	$y^{-1} t_{3i-1} y$	$t_{3i+1} x_i t_{3i} x_i^{-1}$	$z_i^{-1} x_i$

Specifically, when $i = n-2$, we obtain $x_{n-1} = x$. The other generators t_j are recovered from x- and y-conjugations.

For $j < n$, let $(g, h) = (yt_1, z_1 \dots z_{n-2} x z_{n-1-j}^{-1} \dots z_1^{-1})$. Lemma 5.2 implies that $\ell(h) = 2j$, and

$$\begin{aligned} d(T_g, A_h) &= d((z_1 \dots z_{n-1-j}).T_{yt_1}, A_{z_{n-j} \dots z_{n-2} x}) \\ &= d(v_{n-1}, v_1) + d(v_1, v_2) + \cdots + d(v_{n-2-j}, v_{n-1-j}) + d(v_{n-1-j}, v_\star) = 2(n-j) - 1. \end{aligned}$$

For $i \le n-1-j$, let $x_i := z_i \dots z_{n-2} x z_{n-1-j}^{-1} \dots z_i^{-1}$. Since z_i commutes with t_{3i}, a similar relation $z_i = t_{3i+1} x_i t_{3i} x_i^{-1}$ still holds. Therefore, the original generators are recovered as follows:

Generator	y	t_1	x_1	t_{3i-1}	t_{3i}	t_{3i+1}	z_i	x_{i+1}
Recovery	g^4	g^3	h	$x_i t_{3i-2} x_i^{-1}$	$y t_{3i-1} y^{-1}$	$y^{-1} t_{3i-1} y$	$t_{3i+1} x_i t_{3i} x_i^{-1}$	$z_i^{-1} x_i z_i$

For $i > n-1-j$, the recoveries of t_{3i}, t_{3i-1}, t_{3i+1}, z_i, and x_{i+1} are the same as in the case $j = n$.

Action Faithfulness. To show that H_0 is core-free in Γ, we begin with the intersection:

$$H_0 \cap \left(\bigcap_i z_i H_0 z_i^{-1} \right) = \ker(t_4^*, t_7^*, \dots, t_{3n-5}^*).$$

The φ_x^T and φ_y^T actions on those $t_{3i+1}^* \in V_{H_0}^*$ yield the other generators of this dual space. By Lemma 5.1, H_0 is core-free, and the Γ-action on the Bass-Serre tree is faithful.

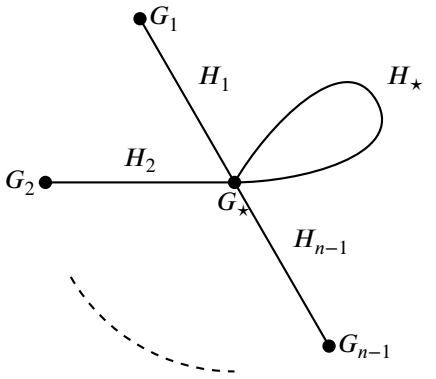
□

A modification of the construction allows the quotient graph to be a star graph with a loop:

Corollary 5.2. *For any $n \ge 3$, each $j = 1, \dots, n$, and every admissible assignment of indices $n_{e,v}$ on a star graph Y of $n-1$ edges and an additional loop at the center, there exists an infinite family of two-generator subgroups $\Gamma = \langle g, h \rangle < \mathrm{Aut}(X_{2n+2})$ with arbitrarily large vertex stabilizers and the following properties:*

- *The generator g is elliptic of order* $\mathrm{ord}(g) = 6$.
- *The generator h is hyperbolic, with $\ell(h) = 2j-1$, and $d(T_g, A_h) = 2(n-j)-1$ when $j < n$, or $d(T_g, A_h) = 0$ when $j = n$.*
- *The minimal quotient graph for Γ is the looped star graph Y, and the associated graph of groups satisfies $[\Gamma_v : \Gamma_e] = n_{e,v}$ for each edge e incident with a vertex v.*

The construction. Let the vertices be $v_\star$, v_1,…, v_{n-1}, and the edges be $e_\star$, e_1,…, e_{n-1}. As before, $v_\star$ is the center vertex, while $e_\star$ is a loop at $v_\star$. We will define the corresponding vertex groups $G_\star$ and G_i, and edge groups $H_\star$ and H_i:



As before, for sufficiently large N, define

$$G_\star = \langle t_0, t_1, \dots, t_{3N-2} \rangle \cong (C_2)^{3N-1}.$$

For $i = 1, \dots, n-2$, define the edge subgroups: if $n_{e_i, v_\star} = 1$, set $H_i = G_\star$. Otherwise,

$$H_{n-1} = \langle t_i \mid i \neq 0 \rangle, \text{ and } H_j = \langle t_i \mid i \neq 3j+2 \rangle,\ j = 1, \dots, n-2.$$

Define $G_{n-1} = H_{n-1} \rtimes \langle y \rangle$, with $\mathrm{ord}(y) = 3$, and the non-trivial conjugation relations on the generators are given as before:

t	t_{3i-1}	t_{3i}	t_{3i+1}	$(i = 1, \dots, N)$
t^y	t_{3i}	t_{3i+1}	t_{3i-1}	

For $j = 1, \dots, n-2$, define $G_j = \langle z_j \rangle \rtimes H_j$, $\mathrm{ord}(z_j) = 3$, with the only non-trivial conjugation relation being

$$t_{3j+1} z_j t_{3j+1} = z_j^2.$$

Finally, define

$$H_\star = \langle t_0, t_1, \dots, t_{3N-3} \rangle,$$

and the boundary injection

$$\varphi : H_\star \to G_\star,\ \varphi(t_i) = x t_i x^{-1} = t_{i+1}.$$

We will show that its fundamental group Γ is two-generated, has the desired geometric quantities, and acts faithfully.

Proof. **Generating Pairs.** For $j = n$, let $(g, h) = (y t_1, y z_1 \dots z_{n-2} x)$. By Lemma 5.2, we have $\ell(h) = 2n-1$, and $d(T_g, A_h) = 0$.

For $j < n$, let $(g, h) = (y t_1, z_1 \dots z_{n-2} x z_{n-1-j}^{-1} \dots z_1^{-1})$. Lemma 5.2 implies that $\ell(h) = 2j-1$, and $d(T_g, A_h) = 2(n-j)-1$.

The conjugation action of the stable letter x on the generators t_i is very similar to that of the elliptic automorphism x in Proposition 5.3. As a result, the recovery follows the same formulas as before.

Action Faithfulness. As in the circuit cases, the subgroup $H_\star$ is core-free, since

$$\bigcap_{i \in \mathbb{Z}} \varphi^i(H_\star) = \bigcap_{i \in \mathbb{Z}} x^i H_\star x^{-1} = \{1\}.$$

□

5.2. Two-generator Subgroups with General Quotient Graphs

The goal of this subsection is to derive a more general graph-of-groups construction:

Proposition 5.4. *Let Y be a finite tree or unicyclic graph, and let $\Delta(Y)$ denote its maximal valency. For every admissible index assignment on Y, there exists an infinite family of two-generator subgroups $\Gamma = \langle g, h\rangle < \mathrm{Aut}(X_{2\Delta(Y)})$ with arbitrarily large vertex stabilizers such that $\ell(h) = l$, $d(T_g, A_h) = d$, the minimal quotient graph for Γ is Y, and the prescribed indices satisfy $|\Gamma_v : \Gamma_e| = n_{e,v}$. In particular, this includes the case when Y is a reduced quotient.*

Let us first introduce a shorthand notation: fix an assignment of numbers $n_{e,v}$, and for $v, v' \in \mathcal{V}(Y)$, denote

$$m_{v,v'} = \#\{k \mid n_{e_i,v_i} > 1\},$$

where $v_0,\dots, v_n$ are consecutive vertices on a path with $v = v_0$ and $v' = v_n$, and e_i is the edge connecting v_i and v_{i+1}.

5.2.1. General Trees

Having discussed the case where Y is a path, we now assume Y is a tree with at least 3 leaves, denoted by $v_0, \dots, v_{n-1}$. Here, Y is obtained by folding the edges of an (l, d)-tadpole graph; we assume that its junction vertex corresponds to $w \in \mathcal{V}(Y)$.

Under the quotient map from the (l, d)-tadpole graph to Y, both the l-circuit and the d-path become walks decomposed into maximal paths. We may assume that the endpoints of these paths lie among $v_0,\dots, v_{n-1}$ and w, since the general case reduces to this one with smaller l and d.

More specifically, we may assume $v_j,\dots, v_{n-1}$ are leaves arising from the circuit, with

$$\sum_{i=j+1}^{n-1} d(v_i, v_{i-1}) + d(v_j, w) + d(v_{n-1}, w) = l,$$

while $v_0,\dots, v_{j-1}$ are leaves arising from the path, with

$$\sum_{i=1}^{j-1} d(v_i, v_{i-1}) + d(v_{j-1}, w) = d.$$

We begin the construction with the path connecting v_0 and v_{n-1}. For $i = 1,\dots, n-2$, let w_i be the projection of v_i to the tree generated by $v_0,\dots, v_{i-1}$, and v_{n-1}. We construct the vertex and edge groups connecting v_i and w_i. In particular, we set $m_i = m_{v_i,w_i}$, and $m_\star = \max_{1\le i\le n-2}(m_{v_0,v_i}, m_{v_{n-1},v_i})$.

The construction. We start with an abelian group generated by involutions,

$$K = \langle t_0, t_1, \dots, t_{3n-6} = p_0, p_1, \dots, p_{2N}, q_1, \dots, q_M\rangle,$$

where

$$M = \sum_{i=1}^{n-2}(m_i - 1), \text{ and } N \ge \frac{M + m_\star}{2}.$$

Define the vertex and edge groups on the path $[v_0, v_{n-1}]$ from K, analogously to the construction in Proposition 5.1: for each vertex $v \in \mathcal{V}([v_0, v_{n-1}])$, define

$$G_v = \langle K, r_{m_{v_{n-1},v_0} - m_{v,v_0}}, \dots, r_{m_{v_{n-1},v_0}-1}, s_1, \dots, s_{m_{v,v_{n-1}}}\rangle,$$

and edge groups H_e similarly, so the indices agree with the numbers $n_{e,v}$. In particular, the edge groups at the leaves v_0 and v_{n-1} are

$$H_0 = \langle K, s_1, \dots, s_{m_{v_0,v_{n-1}}-1}\rangle,$$

and

$$H_{n-1} = \langle K, r_1, \dots, r_{m_{v_{n-1},v_0}-1}\rangle.$$

Define $G_0 = H_0 \rtimes \langle y\rangle$ and $G_{n-1} = H_{n-1} \rtimes \langle x\rangle$, with $\mathrm{ord}(x) = \mathrm{ord}(y) = 3$, and the non-trivial conjugation relations are:

t	t_{3i}	t_{3i+1}	t_{3i+2}	$(i = 0, \dots, n-3)$
t^x	t_{3i+1}	t_{3i+2}	t_{3i}	
t	p_{2i-2}	p_{2i-1}	r_i	$(i = 1, \dots, m_{v_{n-1},v_0} - 1)$
t^x	p_{2i-1}	r_i	p_{2i-2}	
t	p_{2i}	p_{2i+1}		$(i = m_{v_{n-1},v_0} - 1, ..., N-1)$
t^x	p_{2i+1}	$p_{2i}p_{2i+1}$		

t	t_{3i+1}	t_{3i+2}	t_{3i+3}	$(i = 0, \dots, n-3)$
t^y	t_{3i+2}	t_{3i+3}	t_{3i+1}	
t	p_{2i-1}	p_{2i}	s_i	$(i = 1, \dots, m_{v_0,v_{n-1}} - 1)$
t^y	p_{2i}	s_i	p_{2i-1}	
t	p_{2i+1}	p_{2i+2}		$(i = m_{v_0,v_{n-1}} - 1, ..., N-1)$
t^y	p_{2i+2}	$p_{2i+1}p_{2i+2}$		

Next, we construct the vertex and edge groups on the path $[v_i, w_i]$, inductively from $i = 1$ to $i = n-2$. Note that the group G_{w_i} has already been constructed at this stage of the induction, and all these groups will turn out to be abelian groups generated by involutions. Let K_i be the subgroup of G_{w_i} generated by letters except t_{3i-1} and $p_1, \dots, p_{2N}$, and suppose

$$G_{w_i} = \langle K_i, p_1, \dots, p_{n_i}, t_{3i-1} \rangle.$$

In addition, let $n_i' = \sum_{j=1}^{i-1}(m_j - 1)$. For each vertex v on $[v_i, w_i]$, define

$$G_v = \langle K_i, p_1, \dots, p_{n_i - m_{w_i,v}}, o_{n_i'+1}, \dots, o_{n_i'+m_{v,w_i}} \rangle \rtimes \langle t_{3i-1} \rangle,$$

and for each $e = [v, v']$ where $d(w_i, v') - d(w_i, v) = 1$, define

$$H_e = \langle K_i, p_1, \dots, p_{n_i - m_{w_i,v'}}, o_{n_i'+1}, \dots, o_{n_i'+m_{v,w_i}} \rangle \rtimes \langle t_{3i-1} \rangle,$$

where the new generators o_j are involutions, the first factor of the semidirect product is abelian, and the only non-trivial conjugation relation is

$$o_j^{t_{3i-1}} = o_j q_j, \ n_i' + 1 \le j \le n_{i+1}'.$$

Let e_i denote the leaf edge, and define

$$G_{v_i} = \langle H_{e_i}, z_i \rangle = \langle K_i, p_1, \dots, p_{n_i - m_{w_i,v_i}}, o_{n_i'+1}, \dots, o_{n_{i+1}'} \rangle \rtimes \langle t_{3i-1}, z_i \rangle,$$

where $\mathrm{ord}(z_i) = 3$, $\langle t_{3i-1}, z_i \rangle \cong D_6$, and the non-trivial conjugation relations are given by

t	p_j	o_j	q_j	$(j = n_i' + 1, \dots, n_{i+1}')$
t^{z_i}	o_j	$o_j q_j$	$o_j q_j p_j$	
$t^{t_{3i-1}}$	p_j	$o_j q_j$	q_j	

For clarity, we include the conjugation relations for t_{3i-1}, and it is clear that these are compatible with the fact that $\langle t_{3i-1}, z_i \rangle \cong D_6$. Moreover, the construction satisfies the prescribed edge-vertex index assignments. We will show that its fundamental group Γ is two-generated, has the desired geometric quantities, and acts faithfully.

Proposition 5.4, tree case. **Generating Pairs.** If w lies on the tree induced by $v_j, \dots, v_{n-1}$, we may assume it is on the path $[v_j, v_{n-1}]$ by relabeling the leaves in the construction above. The distance sum relation in Lemma 5.2 can be further simplified as

$$\sum_{i=j+1}^{n-1} d(v_i, v_{i-1}) + d(v_j, v_{n-1}) = l.$$

In this scenario, we let $(g, h) = (yt_0, z_1 \dots z_{n-2} x z_{j-1}^{-1} \dots z_1^{-1})$. By Lemma 5.2, the desired geometric quantities hold: $\ell(h) = l$ and $d(T_g, A_h) = d$. We next show that $\Gamma = \langle g, h \rangle$: for the original generators x, y, $t_0, \dots, t_{3n-6} = p_0$, and $z_1, \dots, z_{n-2}$, an argument similar to that in the proof of Proposition 5.3 shows that they are obtained from g and h. The remaining generators, namely p_i, q_i, r_i, s_i, and o_i, are obtained as follows:

Generator	p_{2i-1}	p_{2i}	r_i	s_i	o_j	q_j
Recovery	$xp_{2i-2}x$	$yp_{2i-1}y$	$xp_{2i-1}x$	$yp_{2i}y$	$z_ip_jz_i^{-1}$	$o_jz_io_jz_i^{-1}$

For w not on this induced subtree, the folding condition in the admissible index assignment implies H_e is properly contained in G_w, where e is the edge incident with w in $[v_j, w]$ and $[v_{n-1}, w]$. From our construction, one sees that the generator in $G_w - H_e$ is either s_i or o_i. Let $(g, h) = (yt_0, z_1 \dots z_{n-2}s_ixz_{j-1}^{-1} \dots z_1^{-1})$ (or o_i in place of s_i), so the geometric quantities are satisfied. The generator-recovery argument is similar: s_i commutes with $t_0,\dots, t_{3n-6}$ and $p_1,\dots, p_{2N}$. Although o_i does not commute with t_{3i-1}, a closer examination of the recovery process in Proposition 5.3 indicates that it will not affect the recovery of the original generators when x is replaced with o_ix. The recovery of p_i, q_i, r_i, s_i, and o_i is also similar, and x is recovered once s_i or o_i has been recovered.

Action Faithfulness. To show that H_{n-1} is core-free, we note that the relations

$$t_{3i-1}^{z_i} = t_{3i-1}z_i,\ p_j^{z_i} = o_j,\ q_j^{z_i} = o_jp_jq_j,$$

imply that

$$H_{n-1} \cap z_iH_{n-1}z_i^{-1} \cap z_i^{-1}H_{n-1}z_i = \ker(t_{3i-1}^*, (p_{j_1}^* + q_{j_1}^*), \dots, (p_{j_2}^* + q_{j_2}^*)),$$

where $j_1 = n_i' + 1$ and $j_2 = n_{i+1}'$. In particular, each $p_j^* + q_j^*$, $j = 1, \dots, M$, serves as a null covector for one of the intersections above, for some $1 \le i \le n-2$. In addition, certain compositions of conjugations by x and y send $t_2^*,\dots, t_{3n-7}^*$ to the various t_i^*, p_i^* and r_i^*. Since the covectors t_i^*, p_i^*, $(p_i^* + q_i^*)$ and r_i^* span the entire $V_{H_{n-1}}^*$, H_{n-1} is core-free in Γ.

□

5.2.2. General Unicyclic Graphs

Having discussed the case where Y is a circuit, we now assume Y has at least one leaf vertex. As before, we denote these vertices by $v_0,\dots, v_{n-2}$, and the junction vertex by w. We assume $v_j,\dots, v_{n-2}$ are leaves arising from the circuit, while $v_0,\dots, v_{j-1}$ arise from the path, with distance relations similar to the previous case.

We will begin the construction with the tadpole graph containing the circuit C in Y and the path connecting v_0 to it. For $i = 1,\dots, n-2$, let w_i be the projection of v_i to the tree generated by $v_0,\dots, v_{i-1}$, and C. We construct the vertex and edge groups connecting v_i and w_i. In particular, we set $m_i = m_{v_i,w_i}$, and $m_\star = \max_{1\le i\le n-2}(m_{v_0,v_i})$.

The construction. We start with an abelian group generated by involutions. Let $v_\star$ be the vertex on the circuit of Y that is closest to v_0. As in the construction in Proposition 5.2 and Corollary 5.1, we let m be the number of edge groups on the circuit properly contained in the initial vertex group in either direction. Fix an abelian group generated by involutions,

$$K = \langle t_m, \cdots t_{mN}, q_1, \dots, q_M\rangle,$$

where

$$M = \sum_{i=1}^{n-2}(m_i - 1), \text{ and } N \ge 3n - 4 + m_{v_\star,v_0} + m_\star + M.$$

Define the vertex and edge groups on the circuit $C \subset Y$ from K, analogously to the construction in Proposition 5.2: fix a direction from $v_\star$ to each $v \in \mathcal{V}(C)$, and for each vertex $v \in \mathcal{V}(C)$, define

$$G_v = \langle K, t_{m_{v_\star,v}}, \dots, t_{m-1}, t_{mN+1}, \dots, t_{mN+m_{v,v_\star}}\rangle.$$

In particular,

$$G_{v_\star} = \langle t_0, \cdots t_{mN}, q_1, \dots, q_M\rangle.$$

Define the edge groups H_e similarly; in particular, the last edge subgroup H_e before returning to $G_{v_\star}$ is

$$H_e = \langle t_m, t_{m+1}, \dots, t_{m(N+1)-1}, q_1, \dots, q_M\ \rangle,$$

possibly with an additional $t_{m(N+1)}$, depending on $n_{e,v_\star}$. In both cases, define the boundary monomorphism and HNN stable letter:

$$\varphi : H_e \to G_{v_\star},\ \varphi(t_i) = x t_i x^{-1} = t_{i-m},\ \varphi(q_j) = q_j.$$

Next, we define the vertex and edge groups on the path $[v_0, v_\star]$. Set

$$K_0 = \langle t_0, \dots, t_{(N-m_{v_\star,v_0})m}, t_1, \dots, t_{m-1}, \dots, t_{(N-1)m+1}, \dots, t_{Nm-1}, q_1, \dots, q_M \rangle.$$

For each vertex v on the path, define

$$G_v = \langle K_0, t_{(N-m_{v_\star,v_0}+1)m}, \dots, t_{(N-m_{v_\star,v})m}, s_1, \dots, s_{m_{v,v_\star}} \rangle,$$

and edge groups H_e similarly, so the indices agree with the numbers $n_{e,v}$. In particular, the edge group at the leaf v_0 is

$$H_0 = \langle K_0, s_1, \dots, s_{m_{v_0,v_\star}-1} \rangle.$$

Define $G_0 = H_0 \rtimes \langle y \rangle$, with $\mathrm{ord}(y) = 3$; the non-trivial conjugation relations follow the pattern of Proposition 5.2, Corollary 5.1, and Proposition 5.1:

t	$t_{(3i+1)m}$	$t_{(3i+2)m}$	$t_{(3i+3)m}$	$(i = 0, \dots, n-3)$
t^y	$t_{(3i+2)m}$	$t_{(3i+3)m}$	$t_{(3i+1)m}$	
t	$t_{(3n-5)m+2i}$	$t_{(3n-5)m+2i+1}$		$(1 \le i \le (m-1)/2)$
t^y	$t_{(3n-5)m+2i+1}$	$t_{(3n-5)m+2i} t_{(3n-5)m+2i+1}$		
t	$t_{(3n-4)m+2i-1}$	$t_{(3n-4)m+2i}$		$(1 \le i \le m/2)$
t^y	$t_{(3n-4)m+2i}$	$t_{(3n-4)m+2i-1} t_{(3n-4)m+2i}$		
t	$t_{(3n-4+i)m}$	s_i		$(i = 1, \dots, m_{v_0,v_\star} - 1)$
t^y	s_i	$t_{(3n-4+i)m}$		

Finally, we construct the vertex and edge groups on the path $[v_i, w_i]$. This is very similar to the corresponding construction in the tree case: let K_i be the subgroup of G_{w_i} generated by all letters except $t_{(3i-1)m}$ and t_{jm} for $j \ge 3n-5$, and suppose

$$G_{w_i} = \langle K_i, t_{(3n-5)m}, \dots, t_{n_i m}, t_{(3i-1)m} \rangle.$$

Set n_i' in the same way, and define similarly:

$$G_v = \langle K_i, t_{(3n-5)m}, \dots, t_{(n_i - m_{w_i,v})m}, o_{n_i'+1}, \dots, o_{n_i'+m_{v,w_i}} \rangle \rtimes \langle t_{3i-1} \rangle,$$

$$H_e = \langle K_i, t_{(3n-5)m}, \dots, t_{(n_i - m_{w_i,v'})m}, o_{n_i'+1}, \dots, o_{n_i'+m_{v,w_i}} \rangle \rtimes \langle t_{3i-1} \rangle,$$

and

$$G_{v_i} = \langle H_{e_i}, z_i \rangle = \langle K_i, t_{(3n-5)m}, \dots, t_{(n_i - m_{w_i,v_i})m}, o_{n_i'+1}, \dots, o_{n_{i+1}'} \rangle \rtimes \langle t_{3i-1} \rangle,$$

with conjugation relations

t	p_j	o_j	q_j	$(j = n_i' + 1, \dots, n_{i+1}')$
t^{z_i}	o_j	$o_j q_j$	$o_j q_j p_j$	
$t^{t_{3i-1}}$	p_j	$o_j q_j$	q_j	

Here, we use the shorthand

$$p_j := t_{(3n-4+m_{v_\star,v_0}+j)m}.$$

This gives a graph of groups on Y with the desired indices $n_{e,v}$. We will show that its fundamental group Γ is two-generated and acts faithfully.

Proposition 5.4, unicyclic case. **Generating Pairs.** If w lies on the graph induced by $v_j, \dots, v_{n-2}$ and the circuit, we may assume it is on the path $[v_j, v_{n-1}]$ by relabeling the vertices in the construction above. Let $(g, h) = (yt_0, z_1 \dots z_{n-2} x z_{j-1}^{-1} \dots z_1^{-1})$. By Lemma 5.2, the desired geometric quantities hold: $\ell(h) = l$ and $d(T_g, A_h) = d$. As in the proof of the tree case, the original generators x, y, t_0, t_m,…, $t_{(3n-6)m}$, and z_1,…, z_{n-2} are obtained from g and h. Following the proof of Corollary 5.1, we obtain the other generators t_i from $t_{(3n-6)m}$, x, and y. The recoveries of s_i, p_j, q_j, and o_j are again similar to those in the proof of the tree case.

If w does not lie on this induced subgraph, the folding condition in the admissible index assignment implies there is a generator s_i or o_i in $G_w - H_e$. Let $(g, h) = (yt_0, z_1 \dots z_{n-2} s_i x z_{j-1}^{-1} \dots z_1^{-1})$; the geometric quantities are satisfied, and the original generators are recovered following the same argument as in the tree case.

Action Faithfulness. We aim to show that the group $G_{v_\star}$ is core-free. Note that the HNN stable letter shifts all the generators t_i away; hence

$$\bigcap_i x^i G_{v_\star} x^{-i} = \ker(t_0^*, \dots, t_{mN}^*).$$

As in the proof of the tree case, we also have

$$G_{v_\star} \cap z_i G_{v_\star} z_i^{-1} \cap z_i^{-1} G_{v_\star} z_i = \ker(t_{(3i-1)m}^*, (p_{j_1}^* + q_{j_1}^*), \dots, (p_{j_2}^* + q_{j_2}^*)),$$

where $j_1 = n_i' + 1$ and $j_2 = n_{i+1}'$; each $j = 1, \dots, M$ corresponds to a certain $1 \le i \le n-2$ in the relation above.

Finally, y-conjugations take $t_{(3n-4+j)m}^*$ to s_j^* for $j = 1, \dots, m_{v_0, v_\star} - 1$.

These covectors together span the entire $V_{G_{v_\star}}^*$, so $G_{v_\star}$ is core-free in Γ.

□

6. Reduced Graphs for Two-generator Discrete Subgroups

6.1. Combinatorial Description of the Reduced Graphs

We are now ready to return to Question 1.1. In particular, we fix the ambient tree as the n-regular tree X_n, consider elliptic-hyperbolic pairs $g, h \in \mathrm{Aut}(X_n)$ with fixed geometric quantities $l = \ell(h)$ and $d = d(T_g, A_h)$, and assume that $\Gamma = \langle g, h \rangle$ is discrete and faithful. While a full classification appears impractical, we still seek a description of the graph carrying the group structure of Γ.

The quotient graph arising from the action of Γ on its minimal subtree is not the right object for this purpose: it is not canonical for Γ, since one can naively lengthen the paths in a graph-of-groups description. This trivial manipulation of the graph of groups also allows us to obtain arbitrary underlying graphs from the star graph constructions alone (Proposition 5.3 and 5.2), making the resulting description uninformative.

Instead, we consider the reduced graph of groups associated with Γ as in Definition 2.7, which removes this artificial flexibility. Let us introduce the following shorthand: if the associated group $\mathcal{G}(e)$ of an edge $e \in \mathcal{E}(Y)$ is a proper subgroup of the incident vertex group $\mathcal{G}(v)$, we double the half-edge closer to the vertex v. The reduction of a graph of groups can then be understood through this doubled graph. Namely, if either half of an edge e is not doubled, we may shrink the edge and merge the incident vertices v_1 and v_2. It is clear that the reduction of the other edges incident with v_1 and v_2 follows the rules below. We are ready to state the following description of the possible reduced graphs for two-generator discrete subgroups with given geometric quantities:

Theorem 6.1. *Let X_n be the n-valent tree, and let $l \ge 1$ and $d \ge 0$. There exists a two-generator discrete and faithful subgroup $\Gamma = \langle g, h \rangle$ with geometric quantities $\ell(h) = l$ and $d(T_g, A_h) = d$ whose reduced graph is Y if and only if Y is obtained from the following construction:*

- *Iteratively fold adjacent edges of an (l, d)-tadpole graph so that the resulting graph Y_0 has valencies $\mathrm{val}(v) \le n$ for each vertex.*
- *Apply an admissible index assignment $\{n_{e,v}\}$ to the graph Y_0 such that $\sum_e n_{e,v} \le n$ for each $v \in \mathcal{V}(Y_0)$, representing the assignment by doubled-half-edge shorthand, and reduce Y_0 with double edges according to the rules above. Let the resulting graph be Y.*

In addition, unless $n = 3$ and Y is a path of unit length, there is an infinite family of such two-generator discrete subgroups.

Proof. On the one hand, assume Y_0 is obtained by folding the edges of an (l, d)-tadpole graph, and Y is obtained by reducing Y_0 with an admissible index assignment. By Proposition 5.4, we obtain a two-generator subgroup $\Gamma = \langle g, h\rangle < \mathrm{Aut}(X_n)$ whose minimal quotient graph is Y_0, and whose stabilizer indices match the assignment. Following the reduction rules, the graph of groups (Y_0, Γ) can be reduced to a graph of groups $(Y, \mathcal{G})$ with the desired underlying graph Y.

On the other hand, assume $\Gamma = \langle g, h\rangle < \mathrm{Aut}(X_n)$ is a two-generator discrete subgroup with geometric quantities $l = \ell(h)$ and $d = d(T_g, A_h)$. By Theorem 4.1, the minimal quotient graph Y_0 is the result of folding certain adjacent edges of the (l, d)-tadpole graph. For each edge–vertex incidence $v \in \{\alpha_e, \omega_e\}$ in Y_0, the index $|\Gamma_v : \Gamma_e|$ satisfies the following conditions, analogous to the admissibility conditions:

- Since Γ acts faithfully on its minimal subtree, $|\Gamma_v : \Gamma_e| > 1$ for every leaf vertex v.
- If Y_0 is a path or circuit and the reduction result Y is not a single edge or a loop with identical vertex and edge stabilizers, then at least one non-leaf vertex v and incident edge e is such that $|\Gamma_v : \Gamma_e| > 1$.
- If Y_0 contains a circuit C, then, since the quotient is a graph of finite groups, the orders of the edge and vertex stabilizers satisfy

$$\prod_{e\in\mathcal{E}(C)} |\Gamma_{\alpha_e} : \Gamma_e| = \frac{\prod_{v\in\mathcal{V}(C)}|\Gamma_v|}{\prod_{e\in\mathcal{E}(C)}|\Gamma_e|} = \prod_{e\in\mathcal{E}(C)} |\Gamma_\omega : \Gamma_e|.$$

- If e is the result of folding edges of the (l, d)-tadpole incident with the same vertex v, then there exists an automorphism in Γ that preserves e and maps one incident edge to another, so $|\Gamma_v : \Gamma_v| > 1$.

Although these conditions do not force the actual indices to lie in $\{1, 2, 3\}$, they become the admissibility conditions above after passing to the doubled-half-edge notation. Therefore, the reduced graph of groups $(Y, \mathcal{G})$ follows the description in the theorem.

□

6.2. Inequalities for Reduced Graphs with $d(T_g, A_h) = 0$

Under the additional restriction $d(T_g, A_h) = 0$, the minimal quotient Y_0 and the reduced graph Y are obtained from a circuit rather than a nondegenerate tadpole. This enables us to classify the possible reduced graphs explicitly by inequalities in this setting.

Theorem 6.2. *Let $g, h \in \mathrm{Aut}(X_n)$, where g is elliptic, h is hyperbolic, $d(A_h, T_g) = 0$, and $\Gamma = \langle g, h\rangle$ is a discrete and faithful subgroup. Denote $l = \ell(h)$. Then the reduced graph Y associated with Γ can be any of the following:*

- *If l is even, the reduced graph Y can be any tree satisfying*

$$\sum_{v\in\mathcal{V}(Y)} 2\left\lceil \frac{\mathrm{val}(v) - 1}{n - 2} + \frac{1}{2}\right\rceil \le l + 2.$$

  Except for the case where $l = 2$ and $n = 3$, the order of the vertex groups associated with Y can be arbitrarily large.

- *Whether l is even or odd, the reduced graph Y can be any unicyclic graph satisfying*

$$\sum_{v\in\mathcal{V}(Y)} 2\left\lceil \frac{\mathrm{val}(v) - 1}{n - 2} + \frac{1}{2}\right\rceil - \sum_{v\in\mathcal{V}(C)} \left\lceil \frac{\mathrm{val}(v) - 1}{n - 2} + \frac{1}{2}\right\rceil \le l.$$

  Here C denotes the unique circuit contained in Y. The associated vertex group orders can be arbitrarily large.

Proof. Let $k = \lceil \frac{n}{2}\rceil \ge 2$. For a vertex $v \in \mathcal{V}(Y, \mathcal{G})$, consider the number $\mathrm{s}(v)$ of edges e incident with v for which $\mathcal{G}(e) < \mathcal{G}(v)$:

- If $\mathrm{s}(v) < \mathrm{val}(v) - k$, we say that the vertex v is *deficient*, and assign the weight $\mathrm{w}(v) = \mathrm{val}(v) - 2$ to the vertex.
- If $\mathrm{s}(v) \geq \mathrm{val}(v) - k$, we say that the vertex v is *adequate*, and assign the weight $\mathrm{w}(v) = \mathrm{s}(v) + k - 2$ to the vertex. In particular, if $\mathrm{s}(v) = \mathrm{val}(v)$, we say that the vertex is *robust*.

We have two observations. First, we have $\mathrm{w}(v) \geq \mathrm{val}(v) - 2$ in both cases. Second, $\mathrm{s}(v) = \mathrm{val}(v) - k$ can be viewed as a boundary case: although the vertex is adequate, the equation $\mathrm{w}(v) = \mathrm{val}(v) - 2$ still holds.

When the graph of groups $(Y, \mathcal{G})$ is reduced to $(Y', \mathcal{G}')$ by shrinking an edge e, the vertices v_1 and v_2 incident with e, of valencies n_1 and n_2, merge into a new vertex v of valency $n = n_1 + n_2 - 2$. Let the corresponding numbers of proper boundary monomorphisms be s_1, s_2, and s. If the edge e is single, we have $s_1 \leq n_1 - 1$, $s_2 \leq n_2 - 1$, and $s = s_1 + s_2$. If the half of e near v_2 is doubled, then $s_2 \geq 1$ and $s = n_1 + s_2 - 2$.

Lemma 6.1. *In all cases, we have* $\mathrm{w}(v_1) + \mathrm{w}(v_2) \geq \mathrm{w}(v)$.

Proof. **Case (1): e is single.** If one of the vertices, say v_2, is deficient, then

$$s = s_1 + s_2 \leq n_1 - 1 + n_2 - k - 1 = n - k,$$

implying that v is either deficient or is in the boundary case:

$$\mathrm{w}(v_1) + \mathrm{w}(v_2) \geq (n_1 - 2) + (n_2 - 2) = \mathrm{w}(v).$$

If both v_1 and v_2 are adequate, then either the same argument holds if v is deficient, or

$$\mathrm{w}(v_1) + \mathrm{w}(v_2) = s_1 + s_2 + 2k - 4 \geq s + k - 2 = \mathrm{w}(v)$$

if v is adequate.

Case (2): e is halfway-doubled. Suppose the half of e near v_2 is doubled. If v_2 is deficient, then

$$s = n_1 + s_2 - 2 < n_1 + n_2 - 2 - k = n - k,$$

implying that v is deficient:

$$\mathrm{w}(v_1) + \mathrm{w}(v_2) \geq (n_1 - 2) + (n_2 - 2) = \mathrm{w}(v).$$

If v_2 is adequate, then either the same argument holds if v is deficient, or

$$\mathrm{w}(v_1) + \mathrm{w}(v_2) \geq (n_1 - 2) + (s_2 + k - 2) = (n_1 + s_2 - 2) + k - 2 = s + k - 2 = \mathrm{w}(v),$$

if v is adequate.

□

Now suppose that a discrete subgroup Γ acts on the n-valent tree, and that $(Y_0, \mathcal{G}_0)$ is the quotient graph of groups of the Γ-minimal subtree by Γ, while $(Y, \mathcal{G})$ is a reduced graph of groups obtained from $(Y_0, \mathcal{G}_0)$. Since the covering tree of $(Y_0, \mathcal{G}_0)$ has valency at most n, it follows that $\mathrm{val}(v) + \mathrm{s}(v) \leq n$. It is not difficult to see that $\mathrm{w}(v) \leq n - 2$ for each $v \in \mathcal{V}(Y_0)$, with equality holding if and only if $(\mathrm{val}(v), \mathrm{s}(v)) = (n, 0)$ or $(k, n - k)$.

On the other hand, if a vertex $v \in \mathcal{V}(Y)$ is robust, then we have $\mathrm{w}(v) = \mathrm{val}(v) + k - 2$. Since the weight sum of the graph of groups is non-increasing during the reduction process, v must result from merging at least

$$\left\lceil \frac{\mathrm{val}(v) + k - 2}{n - 2} \right\rceil = \left\lceil \frac{\mathrm{val}(v) - 1}{n - 2} + \frac{1}{2} \right\rceil$$

vertices. Consequently,

$$\sum_{v \in \mathcal{V}(Y)} 2 \left\lceil \frac{\mathrm{val}(v) - 1}{n - 2} + \frac{1}{2} \right\rceil \leq |\mathcal{V}(Y_0)|.$$

If Y_0 is a tree, then Corollary 4.1 implies that

$$|\mathcal{V}(Y_0)| = 1 + |\mathcal{E}(Y_0)| \leq 1 + \frac{l}{2},$$

which gives the desired inequality.

If Y_0 is unicyclic, we sum the vertex-wise inequalities differently by counting the vertices not in the circuit C_0 twice:

$$\sum_{v\in\mathcal{V}(Y)} 2\left\lceil \frac{\mathrm{val}(v)-1}{n-2} + \frac{1}{2}\right\rceil - \sum_{v\in\mathcal{V}(C)} \left\lceil \frac{\mathrm{val}(v)-1}{n-2} + \frac{1}{2}\right\rceil \leq 2|\mathcal{V}(Y_0)| - |\mathcal{V}(C_0')|.$$

Here, C_0' is the set of vertices in Y_0 that ultimately merged into C. Since C_0' contains the circuit C_0, we have

$$2|\mathcal{V}(Y_0)| - |\mathcal{V}(C_0')| \leq 2|\mathcal{V}(Y_0)| - |\mathcal{V}(C_0)| = 2|\mathcal{E}(Y_0)| - \mathrm{girth}(Y_0) \leq l,$$

which gives the desired inequality.

We show that each tree or unicyclic graph Y satisfying the inequality can be realized. Indeed, if $n \geq 4$, we replace a vertex $v \in \mathcal{V}(Y)$ whose valency satisfies $k + (i-1)(n-2) < \mathrm{val}(v) \leq k + i(n-2)$ with a path consisting of a vertex $v_0 \in \mathcal{V}(Y_0)$ with $(\mathrm{val}(v_0), \mathrm{s}(v_0)) = (k, n-k)$ and i vertices $v_1, \ldots, v_i$ with $\mathrm{val}(v_j) \leq n$ and $\mathrm{s}(v_j) = 0$. If $n = 3$, we replace v with a path consisting of two vertices of type $(2,1)$ and $\mathrm{val}(v) - 2$ vertices of type $(3,0)$. If v is on the circuit in Y, then let all the vertices $v_0, \ldots, v_i$ lie on the circuit in Y_0. As shown in the diagram below, our assembly of the vertices is also compatible with the boundary monomorphism indices on the circuit.

The resulting graph Y_0 satisfies the inequality requirement, and by Proposition 5.4, we can construct an elliptic-hyperbolic generated subgroup $\Gamma = \langle g, h\rangle < \mathrm{Aut}(X_n)$ such that $l = \ell(h)$, $d(T_g, A_h) = 0$, and its minimal quotient graph of groups is $(Y_0, \mathcal{G}_0)$, with the indices for the boundary monomorphisms agreeing with the selected $\mathrm{s}(v)$. Thus, $(Y, \mathcal{G})$ is a reduced graph of groups associated with Γ, as claimed.

□

To conclude the discussion from Section 4 through Section 6, we note that Theorem 6.2 has an even more elementary corollary in the trivalent-tree setting. On the one hand, we note that the construction for Theorem 6.2 with $n = 3$ ensures that every non-leaf vertex has type $(2,1)$ or $(3,0)$. Together with the admissible condition $n_{e,v} = 3$ for leaf vertices, this shows that the constructions for Theorem 6.2 and Proposition 5.4 together yield lattices in $\mathrm{Aut}(X_3)$. On the other hand, when $n = 3$, the summation can be simplified:

$$\sum_{v\in\mathcal{V}(Y)} \lceil (\mathrm{val}(v) - 1) + 1/2\rceil = \sum_{v\in\mathcal{V}(Y)} \mathrm{val}(v) = 2|\mathcal{E}(Y)|.$$

These observations imply the following.

Corollary 6.1. *Let $g, h \in \mathrm{Aut}(X_3)$, where g is elliptic, h is hyperbolic, $\ell(h) = l$ and $d(A_h, T_g) = 0$, and $\Gamma = \langle g, h\rangle$ is a lattice. Then the reduced graph Y associated with Γ can be any of the following:*

- *If l is even, the reduced graph Y can be any tree with at most $\frac{l+2}{4}$ edges.*
- *Whether l is even or odd, the reduced graph Y can be any unicyclic graph satisfying*

$$4|\mathcal{E}(Y)| \leq l + 2|\mathcal{E}(C)| + |\pi_0(Y \setminus C)|.$$

Except for the case where $l = 2$ and Y is a tree, the order of the vertex groups can be arbitrarily large.

References

[1] Bass, H., 1993. Covering theory for graphs of groups. J. Pure Appl. Algebra 89, 3–47.

[2] Bestvina, M., 2002. $\mathbb{R}$-trees in topology, geometry, and group theory, in: Daverman, R.J., Sher, R.B. (Eds.), Handbook of Geometric Topology. North-Holland, Amsterdam, pp. 55–91.

[3] Conder, M.D.E., Potočnik, P., 2026. Edge-transitive cubic graphs: analysis, cataloguing and enumeration. J. Algebra 685, 703–737.

[4] Conder, M.J., 2020. Discrete and free two-generated subgroups of SL_2 over non-archimedean local fields. J. Algebra 553, 248–267.

[5] Conder, M.J., 2022. Discrete and free groups acting on locally finite trees. J. Algebra 597, 162–189.
[6] Conder, M.J., Schillewaert, J., 2022. Discrete two-generator subgroups of PSL_2 over non-archimedean local fields. arXiv preprint arXiv:2208.12404 .
[7] Dicks, W., Dunwoody, M.J., 1989. Groups Acting on Graphs. volume 17 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge.
[8] Djoković, D.Ž., 1980. A class of finite group-amalgams. Proc. Amer. Math. Soc. 80, 22–26.
[9] Djoković, D.Ž., Miller, G.L., 1980. Regular groups of automorphisms of cubic graphs. J. Combin. Theory Ser. B 29, 195–230.
[10] Du, Y., Hersonsky, S., 2025. Free and discrete generating pairs of tree automorphisms. arXiv preprint arXiv:2512.00957 .
[11] Gilman, J., 1995. Two-Generator Discrete Subgroups of PSL(2, $\mathbb{R}$). volume 117 of *Mem. Amer. Math. Soc.* American Mathematical Society, Providence, RI.
[12] Goldschmidt, D.M., 1980. Automorphisms of trivalent graphs. Ann. of Math. (2) 111, 377–406.
[13] Gruschko, I.A., 1940. Über die basen eines freien produktes von gruppen. Rec. Math. [Mat. Sbornik] N.S. 8 (50), 169–182.
[14] Kirschmer, M., Rüther, M.G., 2017. The constructive membership problem for discrete two-generator subgroups of $SL_2(\mathbb{R})$. J. Algebra 480, 519–548.
[15] Kurosch, A., 1934. Die untergruppen der freien produkte von beliebigen gruppen. Math. Ann. 109, 647–660.
[16] Markowitz, A., 2025. Recognition and constructive membership for purely hyperbolic groups acting on trees. J. Algebra 663, 375–398.
[17] Nielsen, J., 1955. A basis for subgroups of free groups. Math. Scand. 3, 31–43.
[18] Rosenberger, G., 1986. All generating pairs of all two-generator Fuchsian groups. Arch. Math. (Basel) 46, 198–204.
[19] Serre, J.P., 2002. Trees. Springer-Verlag, Berlin.
[20] Weidmann, R., 2002. The Nielsen method for groups acting on trees. Proc. Lond. Math. Soc. (3) 85, 93–118.

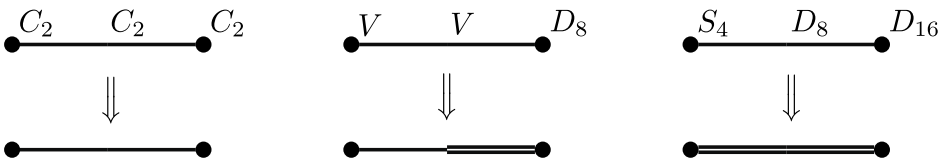


Figure 6.1: The shorthand used to specify when the edge groups are properly contained in the vertex groups. Any edge in a graph of groups falls into one of the three cases illustrated above.

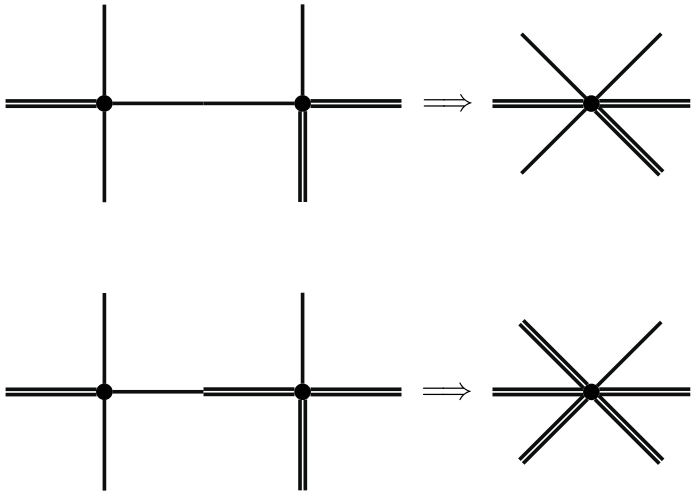


Figure 6.2: If the edge e is single, then after reduction we keep the style of the other edges incident with v_1 and v_2. If the half of e near v_2 is doubled, then after reduction we double the half-edges coming from edges incident with v_1, regardless of their styles before reduction.

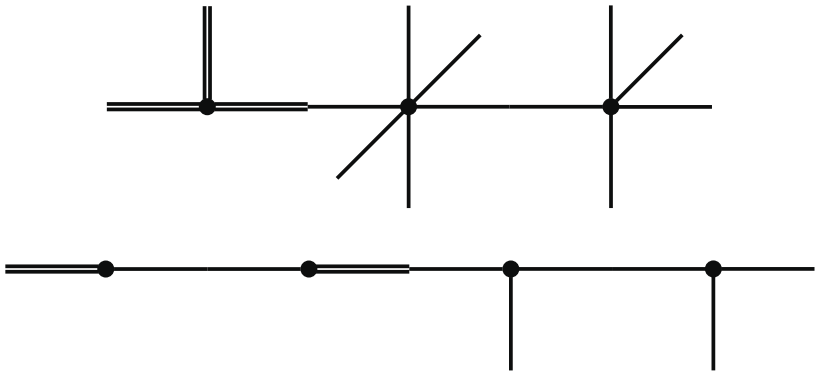


Figure 6.3: Upper diagram: $n = 6$ and $\mathrm{val}(v) = 10$. Bottom diagram: $n = 3$ and $\mathrm{val}(v) = 4$.