PROFINITE NON-RIGIDITY OF ARITHMETIC LATTICES AND THE KÄHLER PROPERTY

YUKUN DU, FENG HAO, AND KEJIA ZHU

Abstract. We prove that the Kählerness of a finitely presented, residually finite group is not determined by its profinite completion, which answers a question asked independently by Arapura and Libgober. In the proof, we give explicit constructions to characterize the absolute Dynkin types that admit non-isomorphic real forms of higher-rank Lie groups containing torsion-free arithmetic lattices with isomorphic profinite completions.

1. Introduction

In 1959, Grothendieck [Gro60] first introduced the algebraic fundamental group to study the fundamental group of varieties and schemes. For a complex algebraic variety X, its algebraic fundamental group $\pi_1^{alg}(X)$ is indeed the profinite completion of $\pi_1(X)$, i.e.

$$\pi_1^{alg}(X) \cong \widehat{\pi_1(X)} := \varprojlim_{N \trianglelefteq_f \pi_1(X)} \pi_1(X)/N.$$

Although the profinite completion of a group records all of its finite quotients, the passage from a discrete group to its profinite completion generally loses information. In the geometric setting, it is natural to ask to what extent $\pi_1(X)$ can be reconstructed from $\widehat{\pi_1(X)}$.

In 1970, Grothendieck [Gro70] posed the following famous question: Let $f : G_1 \to G_2$ be a homomorphism between finitely presented, residually finite groups. If the induced continuous homomorphism $\widehat{f} : \widehat{G}_1 \to \widehat{G}_2$ is an isomorphism, then must f itself be an isomorphism?

Grothendieck's question was answered in the negative by Bridson and Grunewald [BG04]. Moreover, Serre [Ser64] constructed conjugate smooth complex projective varieties X and X^σ such that $\widehat{\pi_1(X)} \cong \widehat{\pi_1(X^\sigma)}$ but $\pi_1(X) \ncong \pi_1(X^\sigma)$.

In Serre's example, X is a smooth projective variety over $\mathbb{C}$, so $\pi_1(X)$ is a Kähler group, namely the fundamental group of a compact Kähler manifold. Thus, Serre's example shows that the profinite completion does not determine a Kähler group up to isomorphism, even among Kähler groups. Nevertheless, it leaves open a more geometric recognition problem. Indeed, the Kähler property imposes strong restrictions not only on a group itself, but also on its finite-index subgroups. Since the profinite completion records the system of finite-index subgroups of a group, it leads to the following fundamental question asked independently by Arapura and Libgober:

Question 1.1. *Is the Kählerness of a residually finite group determined by its profinite completion?*

In the main theorem below, we answer the Question 1.1 negatively (see Theorem 5.8 for the specific construction):

Theorem A. *There exist infinitely many pairs of finitely presented, residually finite groups (G_1, G_2) such that*

$$\widehat{G}_1 \cong \widehat{G}_2,$$

where G_1 is a Kähler group and G_2 is not a Kähler group.

Remark 1.2. *The restriction to residually finite Kähler groups is indispensable. Without residual finiteness, one immediately encounters pathological examples. Indeed, for any Kähler group G and any infinite simple group S, a classical example due to Gromov* [Gro89] *gives $\widehat{G} \cong \widehat{G * S}$, while $G * S$ is not Kähler. The assumption that G' is finitely presented is also natural from a geometric perspective, since every Kähler group is finitely presented.*

Although the main theorem shows that, in general, the Kählerness of a finitely presented, residually finite group is not determined by its profinite completion, in [HLIP$^+$25]*, Hughes, Llosa-Isenrich, Py, Stover, and Vidussi showed that certain groups are determined within the class of residually finite Kähler groups by their profinite completion, including products of surface groups and certain groups with exotic finiteness.*

The proof of Theorem A uses arithmetic lattices with isomorphic profinite completions in different higher-rank semisimple Lie groups. The existence of these lattices is established in Theorem 4.1(1), and an explicit construction is given in Theorem 4.8. The proof of Theorem 4.1 relies on two main ingredients: Lemma 3.4, motivated by the work of Kammeyer and Kionke (see [KK21, KK23]), and the construction of algebraic groups that are isomorphic at every finite place but differ in a prescribed way at certain archimedean places. Allowing multiple noncompact archimedean places enables us to include $\mathrm{Spin}(n,1)$-factors while maintaining the higher-rank condition. Theorem 5.5 then provides the obstruction to Kählerness needed to distinguish the two lattices.

By [KK21],[KK23], the ambient real form is an invariant of the abstract lattice, whereas profinitely isomorphic arithmetic lattices have the same absolute Dynkin type. This raises the question of which non-isomorphic real forms of a fixed absolute Dynkin type admit arithmetic lattices with isomorphic profinite completions. We answer this question in Theorem B, which summarizes Propositions 3.6, 3.10, and 3.13, Theorem 3.15, and Remark 3.16.

Theorem B. *Let G, G' be non-isomorphic higher-rank simply connected real forms of the same absolute type other than E_6. They admit torsion-free cocompact arithmetic lattices with isomorphic profinite completions if and only if they are a pair of compatible indefinite spin, unitary, or quaternionic symplectic groups of type B_ℓ, D_ℓ ($\ell \geq 4$), A_ℓ ($\ell \geq 6$), or C_ℓ ($\ell \geq 6$), real and quaternionic special linear groups of type A_{4n-1} ($n \geq 2$), or the split and minimally non-compact real forms of type E_7 or E_8.*

Remark 1.3. *After our first draft, Py and Isenrich pointed us to Aka's work. In* [Aka12, Theorem 1]*, Aka also constructed pairs of simply-connected absolutely almost simple algebraic groups of the same classical absolute Dynkin type ($D_{(n+1)/2}$ if n is odd and $B_{n/2}$ if n is even) in $\mathrm{Spin}(1,n)^2$ and $\mathrm{Spin}(5,n-4)^2$ for $n \geq 6$. Also, after our first draft, Holger Kammeyer and Adrian Baumann brought their recent work* [BK26] *to our attention. They establish classification results for congruence rigidity of algebraic groups over number fields. Our Theorem B addresses the related question of which absolute Dynkin types admit distinct higher-rank real forms containing arithmetic lattices with isomorphic profinite completions. In Section 3, we give explicit constructions of the arithmetic lattices for each case of Theorem B. These constructions play an essential role in the proof of Theorem A.*

Theorem B, together with Theorem 4.8 (see Theorem 4.1 and Proposition 3.10 for the explicit constructions), can be used to study the profinite invariance of properties related to the ambient real form or the geometry of its symmetric space. We hope it can be useful in the forthcoming work: for example, note that the construction in this paper does not arise from the Grothendieck pair. It is natural to ask if the Grothendieck pair would help determine the Kählerness:

Question 1.4. *Let $f : G_1 \to G_2$ be a homomorphism between finitely presented, residually finite groups. If the induced continuous homomorphism $\widehat{f} : \widehat{G}_1 \to \widehat{G}_2$ is an isomorphism, if G_1 (resp. G_2) is Kähler, then must G_2 (resp. G_1) be Kähler too?*

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2. Preliminaries

We first list some basic definitions, notations, and facts that will be used later. Throughout the paper, k denotes a number field and $\mathcal{O}_k$ its ring of integers. A *place* of k is an equivalence class of nontrivial absolute values on k, where two absolute values are equivalent if they induce the same topology on k. The places of k are divided into *archimedean places* and *finite places*. Write $V_\infty(k)$ and $V_f(k)$ for its sets of archimedean and finite places, respectively. For each place v, let k_v be the completion of k at v. A real place corresponds to an embedding $k \hookrightarrow \mathbb{R}$, and a complex place to a conjugate pair of embeddings $k \hookrightarrow \mathbb{C}$; the corresponding completions are $\mathbb{R}$ and $\mathbb{C}$, respectively. Finite places correspond to nonzero prime ideals of $\mathcal{O}_k$. If v lies above a rational prime p, then k_v is a finite extension of $\mathbb{Q}_p$.

For a linear algebraic k-group $\mathbf{G}$, write $\mathbf{G}_v := \mathbf{G} \times_k k_v$ for its base change to k_v, and $\mathbf{G}(k_v) = \mathbf{G}_v(k_v)$ for its group of k_v-rational points, equipped with its natural topology. We say that two algebraic k-groups $\mathbf{G}$ and $\mathbf{H}$ are isomorphic at v if $\mathbf{G}_v \cong_{k_v} \mathbf{H}_v$.

Using the identification $k \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{v \in V_\infty(k)} k_v$, we have

$$\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) \cong \prod_{v \in V_\infty(k)} \mathbf{G}(k_v).$$

Definition 2.1. *Let $\mathbf{G}$ be a connected reductive algebraic group over a number field k, and let v be a place of k. We say that $\mathbf{G}$ is* k_v-anisotropic *at v if $\operatorname{rank}_{k_v} \mathbf{G} = 0$; otherwise, $\mathbf{G}$ is said to be* k_v-isotropic *at v.*

Definition 2.2. *For a linear algebraic k-group $\mathbf{G}$, fix a faithful k-representation $\rho : \mathbf{G} \to \mathrm{GL}_N$. Define*

$$\mathbf{G}(\mathcal{O}_k) \coloneqq \mathbf{G}(k) \cap \rho^{-1}(\mathrm{GL}(N, \mathcal{O}_k)).$$

We say a subgroup $\Gamma < \mathbf{G}(k)$ is arithmetic *if Γ is commensurable with $\mathbf{G}(\mathcal{O}_k)$. The defined commensurability is independent of the choice of the faithful k-representation ρ.*

2.1. Congruence subgroup property and the finite adelic groups.

Now we list some results from [KK23] regarding the congruence subgroup property and the finite adelic groups that we will cite later in our paper. We first state a theorem originally from Borel and Harish-Chandra [BHC62] but reformulated by Kammeyer and Kionke:

Theorem 2.3. [KK23, Theorem 2.5(a)] *Let k be a number field, let H be a simply connected simple linear algebraic group over k, and let $\Gamma \leq H(k)$ be an arithmetic subgroup. Assume that $H_\infty := \prod_{v \in V_\infty(k)} H(k_v)$ is not compact, where $V_\infty(k)$ is the set of archimedean places of k, then Γ is a lattice in H_∞. Moreover, if $H(k_v)$ is compact for some archimedean place v, then Γ is cocompact.*

Definition 2.4. *For a finite place v, let $\mathcal{O}_v \subset k_v$ be its valuation ring. The ring of finite adeles is the restricted product*

$$\mathbb{A}_k^f := \prod_{v \in V_f(k)}{}' k_v = \{(x_v) : x_v \in k_v \text{ for every } v, \quad x_v \in \mathcal{O}_v \text{ for all but finitely many } v\}.$$

Lemma 2.5. [KK23, Lemma 2.6] *Let k be an algebraic number field and let G, H be two semi-simple linear algebraic groups over k. If G and H are isomorphic at all finite places, i.e., $G \times_k k_v \cong H \times_k k_v$ for all $v \in V_f(k)$, then*

$$G(\mathbb{A}_k^f) \cong H(\mathbb{A}_k^f)$$

as topological groups.

In the following definitions, let k be a number field and $\mathbf{G}$ be a simply connected, absolutely almost simple algebraic group k-group, and let Γ be an arithmetic subgroup of $\mathbf{G}(k)$.

Definition 2.6. *We define the* profinite completion *of* Γ *by*

$$\widehat{\Gamma} := \varprojlim_{N \in \mathcal{N}(\Gamma)} \Gamma/N,$$

where $\mathcal{N}(\Gamma)$ *denotes the set of finite-index normal subgroups of* Γ.

We define the congruence completion *of* Γ *by*

$$\overline{\Gamma} := \varprojlim_{M \in \mathcal{C}(\Gamma)} \Gamma/M,$$

where $\mathcal{C}(\Gamma)$ *denotes the set of normal congruence subgroups of* Γ.

The inclusion $\mathcal{C}(\Gamma) \subset \mathcal{N}(\Gamma)$ *induces a canonical homomorphism* $\widehat{\Gamma} \to \overline{\Gamma}$.

Definition 2.7. *We say that* Γ *has the* congruence subgroup property *(CSP) if the canonical homomorphism,*

$$\widehat{\Gamma} \to \overline{\Gamma},$$

from the profinite completion to the congruence completion has a finite kernel.

We say that $\mathbf{G}(k)$ *has the CSP if some, equivalently every, arithmetic subgroup* $\Gamma < \mathbf{G}(k)$ *has the congruence subgroup property.*

Theorem 2.8. [KK23, Theorem 2.4] *Let* H *be a simply connected, absolutely almost simple algebraic group over a number field* k*. Suppose that* H *has the absolute type:*
B_ℓ *with* $\ell \geq 2$, C_ℓ *with* $\ell \geq 2$, D_ℓ *with* $\ell \geq 5$, E_7, E_8, F_4, G_2, *that* k *is not totally imaginary, and that*

$$\sum_{v \in V_\infty(k)} \operatorname{rank}_{k_v} H \geq 2.$$

Then H *has the congruence subgroup property.*

We next use the adelic consequence of CSP:

Theorem 2.9. [KK23, Theorem 2.5 (b)] *Let* H *be a simply connected simple algebraic group over a number field* k*, and let* $\Delta \leq H(k)$ *be an arithmetic subgroup. Suppose that*

$$H_\infty := \prod_{v \in V_\infty(k)} H(k_v)$$

is noncompact and that H *has CSP. Then there exists an open subgroup* V *of the profinite completion* $\widehat{\Delta}$ *and a compact open subgroup* $K \leq H(\mathbb{A}_k^f)$*, and a topological isomorphism* $V \cong K$.

2.2. **Consequences of Profinitely Isomorphic Lattices.** It is known by Kammeyer–Kionke that profinitely isomorphic lattices have adelic superrigidity:

Theorem 2.10. [KK21, Theorem 3.4] *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be simply connected and absolutely almost simple groups of rank* ≥ 2*, respectively over* k *and* l*. Suppose there are arithmetic subgroups* $\Gamma < \mathbf{G}(k)$ *and* $\Gamma' < \mathbf{H}(l)$ *with* $\widehat{\Gamma} \cong \widehat{\Gamma}'$*. Then there exists an adeles isomorphism* $j : \mathbb{A}_l^f \to \mathbb{A}_k^f$ *together with an isomorphism* $\mathbf{G} \times_k \mathbb{A}_k^f \to \mathbf{H} \times_l \mathbb{A}_l^f$ *over* j.

When $k, l = \mathbb{Q}$, this implies that $\mathbf{G}$ and $\mathbf{H}$ agree at every finite place p, and one has a characterization from the symmetric space of their real places:

Proposition 2.11. [KKRS20, Proposition 2.5] *Let* $\mathbf{G}$ *and* $\mathbf{G}'$ *be semisimple linear algebraic groups over* $\mathbb{Q}$*, such that* $\operatorname{Lie}(\mathbf{G})(\mathbb{Q}_p) \cong \operatorname{Lie}(\mathbf{G}')(\mathbb{Q}_p)$ *for every prime* p*. Then their associated symmetric spaces* $X = \mathbf{G}(\mathbb{R})/K$ *and* $X' = \mathbf{G}'(\mathbb{R})/K'$ *satisfies that* $\dim(X) \equiv \dim(X') \bmod 4$.

Echtler and Kammeyer provide another necessary condition for profinitely isomorphic lattices in terms of their real places.

Definition 2.12. *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be algebraic groups over a field* F *which become isomorphic over the algebraic closure* $\overline{F}$*. They are said to be* inner forms *of one another if there exists an isomorphism* $\varphi : \mathbf{G}_{\overline{F}} \xrightarrow{\sim} \mathbf{H}_{\overline{F}}$ *such that for every* $\sigma \in \mathrm{Gal}(\overline{F}/F)$*, the automorphism* $\varphi^{-1} \circ {}^{\sigma}\varphi : \mathbf{G}_{\overline{F}} \to \mathbf{G}_{\overline{F}}$ *is in* $\mathrm{Inn}(G_{\overline{F}})$*.*

Proposition 2.13. [EK24, Proposition 8] *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be simply connected and absolutely almost simple groups of rank* ≥ 2*, respectively over* k *and* l*. Suppose* $\mathbf{G}(k)$ *has a unique isotropic archimedean place* v*, and* $\mathbf{H}(l)$ *has a unique isotropic archimedean place* w*. If there are arithmetic subgroups* $\Gamma < \mathbf{G}(k)$ *and* $\Gamma' < \mathbf{H}(l)$ *with* $\widehat{\Gamma} \cong \widehat{\Gamma'}$*, then* $\mathbf{G}(k_v)$ *and* $\mathbf{H}(l_w)$ *are inner twists of each other.*

2.3. **Central simple algebras and Brauer classes.** Let A be a *central simple algebra* over k, that is a finite-dimensional associative simple k-algebra with center $Z(A) = k$. In particular, a 4-dimensional central simple algebra A/k is isomorphic to a quaternion algebra over k.

By the Artin–Wedderburn theorem, there exists a unique central division algebra D/k, up to isomorphism, and a positive integer r such that $A \cong M_r(D)$.

Definition 2.14. *Two central simple algebras* A/k*,* A'/k *are* Brauer equivalent *if their underlying central division algebras* D/k *and* D'/k *are isomorphic. We denote by* $[A]$ *the* Brauer class *of* A*.*

The set of Brauer classes forms a group, under the tensor product operation $([A_1], [A_2]) \to [A_1 \otimes_k A_2]$*, called the* Brauer group *of* k *and is denoted by* $\mathrm{Br}(k)$*.*

For a place v of k, the local Brauer group $\mathrm{Br}(k_v)$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$ if v is non-archimedean, is isomorphic to $\frac{1}{2}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ if v is real, and is trivial if v is complex.

Theorem 2.15 (Albert–Brauer–Hasse–Noether)**.** *For a place* v *of* k*, denote the corresponding local invariant map by* $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$*. Then for every central simple algebra* A/k*, one has*

$$\sum_v \mathrm{inv}_v([A \otimes_k k_v]) = 0 \in \mathbb{Q}/\mathbb{Z}.$$

3. On the Profinite Rigidity of Simple Lie Group Lattices

In this section, we prove Theorem B (see Propositions 3.6, 3.10, and 3.13, and Theorem 3.15 for more details). By Theorem 2.10, profinitely isomorphic arithmetic lattices in algebraic groups of rank ≥ 2 induce a bijection between the finite places of their number fields, under which the corresponding local fields and algebraic groups are isomorphic. After base change to algebraic closures, these local isomorphisms yield an isomorphism of the absolute root systems of the ambient algebraic groups. On the other hand, we are concerned with whether different Lie groups of the same absolute Lie type can possess lattices with isomorphic profinite completions.

3.1. **An exact isomorphism after passing to finite index.** The following two facts (see Lemma 3.1 and Lemma 3.2) are standard, we include the proof for the sake of completeness:

Lemma 3.1. *Let* Γ *be a residually finite group and let* $H \leq \Gamma$ *be a finite-index subgroup, then the closure* $\overline{H}$ *of* H *in* $\widehat{\Gamma}$ *is open and that* $\Gamma \cap \overline{H} = H$*.*

Proof. Define $C := \bigcap_{\gamma \in \Gamma} \gamma H \gamma^{-1}$. Note C is actually the kernel of the action of Γ on the finite set Γ/H, thus C is a finite-index normal subgroup of Γ. Since Γ/C is finite, it is a discrete profinite group. Now by the universal property of profinite groups, $q : \Gamma \longrightarrow \Gamma/C$ extends uniquely to a continuous surjection $\widehat{q} : \widehat{\Gamma} \longrightarrow \Gamma/C$. Moreover, $W_H := \widehat{q}^{-1}(H/C)$ is an open and closed subgroup of $\widehat{\Gamma}$. It's easy to see $\Gamma \cap W_H = q^{-1}(H/C) = H$. Because Γ is dense in $\widehat{\Gamma}$ and W_H is open, the intersection $\Gamma \cap W_H = H$ is dense in W_H. On the other hand, W_H is closed and contains H. It follows that $\overline{H} = W_H$. So we can conclude that $\overline{H} = W_H$ is open in $\widehat{\Gamma}$ and $\Gamma \cap \overline{H} = \Gamma \cap W_H = H$. □

Lemma 3.2. *Let* Γ *be a residually finite group and* $V \subset \widehat{\Gamma}$ *is open, then* $G := \Gamma \cap V$ *has finite index in* Γ*.*

Proof. Since V is open in the compact (topological) group $\widehat{\Gamma}$, it has finite index. Consider the map of coset spaces

$$\phi : \Gamma/G \longrightarrow \widehat{\Gamma}/V, \qquad \gamma G \longmapsto \gamma V.$$

Note ϕ is injective: if $\gamma_1 V = \gamma_2 V$, then $\gamma_2^{-1}\gamma_1 \in V \cap \Gamma = G$, and hence $\gamma_1 G = \gamma_2 G$. ϕ is also surjective. Indeed, every coset xV is open in $\widehat{\Gamma}$, so the density of Γ implies that xV contains some $\gamma \in \Gamma$. Then $xV = \gamma V$. Thus $[\Gamma : G] = [\widehat{\Gamma} : V] < \infty$. □

Proposition 3.3. *Let Γ_1 and Γ_2 be residually finite groups. Suppose that there are isomorphic open subgroups $U_i \leq \widehat{\Gamma}_i$. If each Γ_i has a torsion-free, finite-index subgroup, then there are torsion-free, finite-index subgroups $G_i \leq \Gamma_i$ such that $\widehat{G}_1 \cong \widehat{G}_2$.*

Proof. Let $\varphi : U_1 \to U_2$ be an isomorphism. Choose torsion-free finite-index subgroups $T_i \leq \Gamma_i$ and denote their closures in $\widehat{\Gamma}_i$ by $\overline{T}_i$. Since T_i has finite index in Γ_i (thus also residually finite), by Lemma 3.1, $\overline{T}_i$ is open in $\widehat{\Gamma}_i$ and $\Gamma_i \cap \overline{T}_i = T_i$. Set

$$V_1 = U_1 \cap \overline{T}_1 \cap \varphi^{-1}(U_2 \cap \overline{T}_2), \qquad V_2 = \varphi(V_1).$$

Then V_i is open in $\widehat{\Gamma}_i$ (hence V_i is profinite) and φ restricts to an isomorphism $V_1 \cong V_2$. Define $G_i := \Gamma_i \cap V_i$. By Lemma 3.2, each G_i has finite index in Γ_i. Also note $G_i \leq T_i$, hence it is torsion-free. Moreover, Γ_i is dense in $\widehat{\Gamma}_i$ and V_i is open, $G_i = \Gamma_i \cap V_i$ is dense in V_i, thus $\overline{G_i} = V_i$.

Now since $G_i = \Gamma_i \cap V_i$ has finite index in Γ_i, it's easy to see that the profinite topology on G_i induced from Γ_i (and thus from V_i) is the full profinite topology of G_i. Consequently, completion of the inclusion to the profinite group V_i:$G_i \hookrightarrow V_i$ gives a topological inclusion: $\widehat{G}_i = \overline{G}_i \hookrightarrow V_i$, which is actually also a surjection since G_i is dense in V_i, thus we get a topological isomorphism $\widehat{G}_i \cong V_i$. Therefore $\widehat{G}_1 \cong V_1 \cong V_2 \cong \widehat{G}_2$. □

Lemma 3.4. *Let k be a number field, and let $\mathbf{G}_i$, $i = 1, 2$ be two non-isomorphic simply connected and absolutely almost simple k-groups. Suppose that for $i = 1, 2$, $\mathbf{G}_i(k)$ has the* congruence subgroup property, *$G_{i,\infty} := \prod_{v \in V_\infty(k)} \mathbf{G}_i(k_v)$ is noncompact, and for any finite place $v \in V_f(k)$, the completions*

$$\mathbf{G}_1 \times_k k_v \cong \mathbf{G}_2 \times_k k_v.$$

Then there exists torsion-free arithmetic lattices

$$\Gamma_1 < \mathbf{G}_1(k), \ \Gamma_2 < \mathbf{G}_2(k),$$

such that the profinite completions $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$.

Proof. By assumption, for every finite place v of k, $\mathbf{G}_1 \times_k k_v \cong \mathbf{G}_2 \times_k k_v$. By Lemma 2.5, there is an isomorphism of topological groups $\Phi\colon \mathbf{G}_1(\mathbb{A}_k^f) \cong \mathbf{G}_2(\mathbb{A}_k^f)$. Let $\Delta_i \leq \mathbf{G}_i(k), i = 1, 2$ be arithmetic subgroups; for example, we can take $\Delta_i = \mathbf{G}_i(\mathcal{O}_k)$. Since $G_{i,\infty}$ is noncompact and $\mathbf{G}_i$ has the congruence subgroup property, by Theorem 2.9, there exists an open subgroup $V_i \leq \widehat{\Delta}_i$, and a compact open subgroup $C_i \leq \mathbf{G}_i(\mathbb{A}_k^f)$, such that there is a topological isomorphism $\theta_i\colon V_i \cong C_i$ for each $i = 1, 2$.

Define $L := \Phi(C_1) \cap C_2$. Since $\Phi(C_1)$ and C_2 are compact open subgroups of $\mathbf{G}_2(\mathbb{A}_k^f)$, their intersection L is a compact open subgroup. Moreover, L is open in both $\Phi(C_1)$ and C_2. Define

$$U_1 := \theta_1^{-1}(\Phi^{-1}(L)) \leq V_1, \qquad U_2 := \theta_2^{-1}(L) \leq V_2.$$

The subgroup $\Phi^{-1}(L)$ is open in C_1, and L is open in C_2. It follows that U_i is open in V_i, thus open in $\widehat{\Delta}_i$. The composition $\psi := \theta_2^{-1} \circ \Phi \circ \theta_1$ restricts to a topological isomorphism $\psi\colon U_1 \xrightarrow{\sim} U_2$. Thus $\widehat{\Delta}_1$ and $\widehat{\Delta}_2$ contain isomorphic open subgroups.

The arithmetic groups Δ_1 and Δ_2 are residually finite by Mal'cev's theorem and have torsion-free finite-index subgroups by Selberg's lemma. Proposition 3.3 therefore applies and gives torsion-free finite-index subgroups $\Gamma_i \leq \Delta_i, i = 1, 2$, such that $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$. Since finite-index subgroups of arithmetic groups are arithmetic, each Γ_i is an arithmetic subgroup of $\mathbf{G}_i(k)$. □

Corollary 3.5. *For $i = 1, 2$, write $G_{i,\infty} = \prod_{v \in V_\infty(k)} \mathbf{G}_i(k_v)$ as in Lemma 3.4 as a product of compact and non-compact factors:*

$$G_{i,\infty} = G_{i,\infty}^{\mathrm{nc}} \times G_{i,\infty}^{\mathrm{c}}.$$

Then Γ_i is isomorphic to a irreducible lattice $\overline{\Gamma}_i$ in $G_{i,\infty}^{\mathrm{nc}}$; in particular, $\widehat{\overline{\Gamma}_1} \cong \widehat{\overline{\Gamma}_2}$.

Moreover, if $G_{i,\infty}^{\mathrm{c}}$ is non-trivial, then $\overline{\Gamma}_i$ is cocompact.

Proof. By the Theorem 2.3, the noncompactness of $G_{i,\infty}$ implies that Δ_i, and hence Γ_i, is a lattice in $G_{i,\infty}$. If $\mathbf{G}_i(k_v)$ is compact for some archimedean place v, then the same theorem implies that this lattice is cocompact.

Projection onto $G_{i,\infty}^{\mathrm{nc}}$ sends Γ_i to a lattice. Its kernel $\Gamma_i \cap G_{i,\infty}^{\mathrm{c}}$ is finite, since Γ_i is discrete and $G_{i,\infty}^{\mathrm{c}}$ is compact. Since Γ_i is torsion-free, the kernel is trivial. Thus Γ_i is naturally isomorphic to its image in $G_{i,\infty}^{\mathrm{nc}}$.

Since $\mathbf{G}_i$ is absolutely almost simple over k, the standard irreducibility theorem for arithmetic lattices implies that the image of Γ_i in $G_{i,\infty}^{\mathrm{nc}}$ is irreducible; see [LS03, Section 7.2]. □

3.2. **Profinite Non-rigidity of Simple Lie Groups.** In the following sub-subsections, we construct families of pairs of simply connected absolutely almost simple k-groups $\mathbf{G}$ and $\mathbf{G}'$ of the same absolute Dynkin type A_ℓ, B_ℓ, C_ℓ, or D_ℓ such that $\mathbf{G}_v \cong \mathbf{G}'_v$ for every finite place $v \in V_f(k)$, while $\mathbf{G}_w \ncong \mathbf{G}'_w$ for some real place $w \in V_\infty(k)$. Applying Lemma 3.4 and Corollary 3.5, these constructions yield arithmetic lattices in different archimedean forms with isomorphic profinite completions. Thus the profinite completion of an arithmetic lattice does not, in general, determine the archimedean form of its ambient algebraic group.

Throughout this subsection, unless otherwise stated, let α be an algebraic unit such that $k = \mathbb{Q}(\alpha)$ is totally real, with $r_+ \geq 1$ real embeddings $v_1, \ldots, v_{r_+}$ at which it is positive, and exactly one distinguished real embedding at which α is negative. The choice of α may vary between constructions.

3.2.1. *Classical Real Form Constructions.*

Proposition 3.6. *There exist torsion-free cocompact arithmetic lattices $\Gamma < G$, $\Gamma' < G'$ with profinite completions $\widehat{\Gamma} \cong \widehat{\Gamma}'$ satifying any of the following:*

- *$G = \mathrm{Spin}(m, n)$, $G' = \mathrm{Spin}(m', n')$, where $d = m + n = m' + n' \geq 8$, $m, n, m', n' \geq 2$, and $mn \equiv m'n' \bmod 4$;*
- *$G = \mathrm{SU}(m, n)$, $G' = \mathrm{SU}(m', n')$, where $d = m + n = m' + n' \geq 7$, $m, n, m', n' \geq 2$, and $mn \equiv m'n' \bmod 2$;*
- *$G = \mathrm{Sp}(m, n)$, $G' = \mathrm{Sp}(m', n')$, where $d = m + n = m' + n' \geq 6$ and $m, n, m', n' \geq 2$.*

Remark 3.7. *The algebraic groups used in the three constructions above are simply connected and absolutely almost simple of classical absolute Dynkin type. If $d = m + n = m' + n'$, then the spin groups have absolute type B_ℓ when $d = 2\ell + 1$ and D_ℓ when $d = 2\ell$; the special unitary groups have absolute type A_{d-1}; and the quaternionic unitary groups giving the real forms $\mathrm{Sp}(m, n)$ have absolute type C_d. These identifications, together with the dimensions of the associated symmetric spaces, are summarized in Table 1 in Appendix A. After projecting away the compact archimedean factors, the ambient Lie groups are precisely the groups denoted $G_{i,\infty}^{\mathrm{nc}}$ in Corollary 3.5.*

We first construct the defining forms and verify that the associated algebraic groups are isomorphic at every finite place:

Lemma 3.8. • *Suppose $m, n, m', n' \geq 2$, $m + n = m' + n' = d$, and $4 \mid (m - m')$. Define*

$$q = \langle 1^{[m]}, \alpha^{[n]} \rangle, \quad q' = \langle 1^{[m']}, \alpha^{[n']} \rangle,$$

and let $\mathbf{G} = \mathrm{Spin}(q)$ and $\mathbf{G}' = \mathrm{Spin}(q')$ be the associated spin k-groups. Then for every finite place v of k,

$$\mathbf{G} \times_k k_v \cong \mathbf{G}' \times_k k_v.$$

- *Suppose a unit $a \in k$ satisfies that $\sigma_v(a) < 0$ at every real place (e.g., $a = -1$), $E = k(\sqrt{a})$, $m, n, m', n' \geq 2$, $m + n = m' + n' = d$, and $2 \mid (m - m')$. Define the E/k-Hermitian forms*
$$h = \langle 1^{[m]}, \alpha^{[n]} \rangle, \quad h' = \langle 1^{[m']}, \alpha^{[n']} \rangle,$$
and let $\mathbf{G} = \mathrm{SU}(h, E/k)$ and $\mathbf{G}' = \mathrm{SU}(h', E/k)$. Then for every finite place v of k,
$$\mathbf{G} \times_k k_v \cong \mathbf{G}' \times_k k_v.$$
- *Suppose units $a, b \in k$ satisfies that $\sigma_v(a), \sigma_v(b) < 0$ at every real place (e.g., $a = b = -1$), $B = (a, b)_k$ be a quaternion algebra, $m, n, m', n' \geq 2$, and $m + n = m' + n' = d$. Define the quaternionic Hermitian forms*
$$h = \langle 1^{[m]}, \alpha^{[n]} \rangle, \quad h' = \langle 1^{[m']}, \alpha^{[n']} \rangle,$$
and let $\mathbf{G} = \mathrm{Sp}(h, B/k)$ and $\mathbf{G}' = \mathrm{Sp}(h', B/k)$. Then for every finite place v of k,
$$\mathbf{G} \times_k k_v \cong \mathbf{G}' \times_k k_v.$$

Proof. **Case (1): spin groups.** By [Lam05, Chapter VI, Theorem 2.12], a nondegenerate quadratic form over a non-archimedean local field is determined up to isometry by its dimension, determinant square class, and Hasse invariant. Let v be a finite place of k. For $q = \langle 1^{[m]}, \alpha^{[n]} \rangle$, the determinant square class is
$$[\det(q)]_v = [\alpha^n]_v \in k_v^\times / (k_v^\times)^2,$$
and the Hasse invariant is
$$\epsilon_v(q) = \prod_{i<j} (a_i, a_j)_v = (\alpha, \alpha)_v^{n(n-1)/2},$$
because $(1, 1)_v = (1, \alpha)_v = 1$.

Since $4 \mid (m - m')$, $4 \mid (n - n')$, hence q and q' have the same determinant square class and the same Hasse invariant at every finite place, so $q \otimes_k k_v \cong q' \otimes_k k_v$. Therefore, the associated spin groups are k_v-isomorphic.

Case (2): unitary groups. If v splits in E/k, then $E_v \cong k_v \times k_v$, and
$$\mathrm{SU}(h, E_v/k_v) \cong \mathrm{SL}(d, k_v),$$
independently of the Hermitian form h.

Suppose that v is nonsplit, so E_v/k_v is a quadratic field extension. Hermitian forms over E_v/k_v are classified by their dimension and determinant class in
$$k_v^\times / N_{E_v/k_v}(E_v^\times).$$
Since $2 \mid (m - m')$, $n - n'$ is also even, so
$$\frac{\det(h)}{\det(h')} = \alpha^{n-n'}$$
is a square in $k_v^\times$, hence a norm from $E_v^\times$. Thus h and h' have the same local determinant invariant, and the corresponding special unitary groups are k_v-isomorphic.

Case (3): symplectic groups. If B_v is split, then $B_v \cong M_2(k_v)$, and Morita equivalence gives
$$\mathrm{Sp}(h, B_v/k_v) \cong \mathrm{Sp}(2d, k_v),$$
independently of h.

If B_v is division, the reduced norm map
$$\mathrm{Nrd} : B_v^\times \longrightarrow k_v^\times$$
is surjective. Hence, all nondegenerate Hermitian forms of a fixed dimension over B_v with respect to the canonical involution are isometric. In particular $h \otimes_k k_v \cong h' \otimes_k k_v$, and the corresponding k_v-groups are isomorphic. □

The remaining spin cases require local similarities rather than isometries. In one of these cases, we also modify a diagonal coefficient by a totally positive element:

Lemma 3.9. *Let $d \geq 6$ be even, and let $m, n, m', n' \geq 2$ satisfy*

$$m+n = m'+n' = d, \qquad m \equiv n \pmod 4, \qquad m' \equiv n' \pmod 4, \qquad m - m' \equiv 2 \pmod 4.$$

- *If $d/2$ is odd, there exist α so for the quadratic forms*

$$q = \langle 1^{[m]}, \alpha^{[n]} \rangle, \quad q' = \langle 1^{[m']}, \alpha^{[n']} \rangle,$$

and $\mathbf{G} = \mathrm{Spin}(q)$ and $\mathbf{G}' = \mathrm{Spin}(q')$,

$$\mathbf{G} \times_k k_v \cong \mathbf{G}' \times_k k_v,$$

for every finite place v of k.
- *If $d/2$ is even, there exist α and β, β is positive at every real place, so for the quadratic forms*

$$q = \langle 1^{[m-1]}, \beta, \alpha^{[n]} \rangle, \quad q' = \langle 1^{[m'-1]}, \beta, \alpha^{[n']} \rangle,$$

and $\mathbf{G} = \mathrm{Spin}(q)$ and $\mathbf{G}' = \mathrm{Spin}(q')$,

$$\mathbf{G} \times_k k_v \cong \mathbf{G}' \times_k k_v,$$

for every finite place v of k.

Proof. For both cases, we assume that the minimal polynomial of α is irreducible modulo 2, so that k has a unique place v_2 above 2.

Case (1): $d/2$ is odd. Assume in addition that $[k : \mathbb{Q}]$ is even. At every finite place not above 2, an argument similar to Lemma 3.8 gives the required local isomorphism. At v_2, Hilbert reciprocity gives $(\alpha, \alpha)_{v_2} = -1$. For $\lambda \in k_{v_2}^\times$, bilinearity of the Hilbert symbol gives

$$\begin{aligned} \epsilon_{v_2}(\lambda q) &= (\lambda, \lambda)_{v_2}^{m(m-1)/2} (\lambda, \lambda\alpha)_{v_2}^{mn} (\lambda\alpha, \lambda\alpha)_{v_2}^{n(n-1)/2} \\ &= (\lambda, -\alpha)_{v_2}\, \epsilon_{v_2}(q). \end{aligned}$$

Since $[k : \mathbb{Q}]$ is even and α is negative at exactly one real place, the number of real places at which $-\alpha$ is negative is odd. Hilbert reciprocity therefore gives

$$(-\alpha, -\alpha)_{v_2} = -1,$$

so $-\alpha$ is not a square in k_{v_2}. Hence there exists $\lambda \in k_{v_2}^\times$ with $(\lambda, -\alpha)_{v_2} = -1$. It follows that

$$\epsilon_{v_2}(\lambda q) = -\epsilon_{v_2}(q) = \epsilon_{v_2}(q').$$

Because d is even, scaling by λ does not change the determinant square class, and therefore $\lambda q \cong q'$ over k_{v_2}. Since scalar multiples define the same special orthogonal and spin groups, the two spin groups are locally isomorphic at v_2 as required.

Case (2): $d/2$ is even. Assume β is not a square in k_{v_2}. Such a choice of α and β is possible; for example, take α to be a root of

$$f(x) = x^3 - 3x^2 - 2x + 1$$

and put $\beta = \alpha + 1$. The reduction of f modulo 2 is irreducible. A direct comparison modulo 4, using $\alpha^3 = 3\alpha^2 + 2\alpha - 1$, shows that β is not a square in k_{v_2}.

Since β is totally positive, these forms have signatures (m, n) and (m', n') at the distinguished real place and are positive definite at every other real place. Their determinant square classes agree at every finite place. For a finite place $v \nmid 2$,

$$\epsilon_v(q) = (\alpha, \beta)_v^n (\alpha, \alpha)_v^{n(n-1)/2}, \qquad \epsilon_v(q') = (\alpha, \beta)_v^{n'} (\alpha, \alpha)_v^{n'(n'-1)/2}.$$

Here $(\alpha, \alpha)_v = 1$, and $n - n'$ is even, so the Hasse invariants agree. At v_2, for $\lambda \in k_{v_2}^\times$ one similarly obtains

$$\epsilon_{v_2}(\lambda q) = (\lambda, \beta)_{v_2} \epsilon_{v_2}(q) = -(\lambda, \beta)_{v_2} \epsilon_{v_2}(q').$$

Since β is not a square, choose λ with $(\lambda, \beta)_{v_2} = -1$. Then $\lambda q \cong q'$ over k_{v_2}, so the associated spin groups are locally isomorphic at every finite place. □

Proof of Proposition 3.6. Since $m+n = m'+n' = d$, we have

$$mn - m'n' = m(d-m) - m'(d-m') = (m-m')(d-m-m') = (m-m')(n-m').$$

Up to interchanging m and n, which preserves the isomorphic type of G, the modulo condition in case (1) implies either $4 \mid (m-m')$, or $m - m' \equiv n - m' \equiv 2 \bmod 4$. The modulo condition in case (2) implies $2 \mid (m-m')$.

Case (1): spin groups. In the two subcases, Lemma 3.8, case (1), and Lemma 3.9 provide algebraic groups $\mathbf{G}$ and $\mathbf{G}'$ that are isomorphic at every finite place. As spin groups of quadratic forms of dimension at least 5 and have real rank ≥ 2, Kneser's congruence subgroup theorem [Kne79] implies that both $\mathbf{G}$ and $\mathbf{G}'$ have CSP. Lemma 3.4 therefore yields torsion-free arithmetic subgroups

$$\widetilde{\Gamma} < \mathbf{G}(k), \qquad \widetilde{\Gamma}' < \mathbf{G}'(k),$$

with isomorphic profinite completions.

By the Borel–Harish-Chandra theorem (Theorem 2.3), $\widetilde{\Gamma}$ is a lattice in

$$\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) \cong \prod_{v \in V_\infty(k)} \mathrm{Spin}(q \otimes_k k_v).$$

At a real place v with $\sigma_v(\alpha) > 0$, the form q is positive definite and the corresponding factor is $\mathrm{Spin}(d)$. At the distinguished real place with $\sigma_v(\alpha) < 0$, the factor is $\mathrm{Spin}(m,n)$. Counting the number of different type of real places, we obtain

$$\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) \cong \mathrm{Spin}(d)^{r_+} \times \mathrm{Spin}(m,n).$$

Project away from the compact factor $\mathrm{Spin}(d)^{r_+}$ and put

$$\Gamma = \mathrm{pr}(\widetilde{\Gamma}).$$

The kernel of $\mathrm{pr}\,|_{\widetilde{\Gamma}}$ is finite because it is a discrete subgroup of a compact group. Since $\widetilde{\Gamma}$ is torsion-free, this kernel is trivial. Thus $\Gamma \cong \widetilde{\Gamma}$. Because $r_+ \geq 1$, Corollary 3.5 gives cocompactness and irreducibility. The same argument for $\mathbf{G}'$ produces $\Gamma' \cong \widetilde{\Gamma}'$. Therefore $\widehat{\Gamma} \cong \widehat{\Gamma}'$.

Case (2): unitary groups. Lemma 3.8, case (2) provides algebraic groups $\mathbf{G}$ and $\mathbf{G}'$ that are isomorphic at every finite place. The groups $\mathbf{G}$ and $\mathbf{G}'$ are simply connected absolutely almost simple outer k-groups of absolute type A_{d-1}. Since k is not totally imaginary and the groups are assumed to have archimedean rank at least 2, Kammeyer and Kionke's congruence subgroup theorem [KK23, Theorem 2.4] applies to these special unitary groups of type ${}^2A_{d-1}$. Lemmas 3.4 therefore give torsion-free arithmetic lattices in $\mathbf{G}(k)$ and $\mathbf{G}'(k)$ with isomorphic profinite completions.

Since $\sigma_v(a) < 0$ at every archimedean place, the quadratic algebra $E_v \cong \mathbb{C}$. The archimedean factor in the Borel-Harish-Chandra theorem is $\mathrm{SU}(m,n)$ when $\sigma_v(\alpha) < 0$ and the compact group $\mathrm{SU}(d)$ when $\sigma_v(\alpha) > 0$. Consequently,

$$\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) \cong \mathrm{SU}(d)^{r_+} \times \mathrm{SU}(m,n).$$

The same formula holds for $\mathbf{G}'$ with (m,n) replaced by (m',n'). Projecting away the compact $\mathrm{SU}(d)^{r_+}$ factors is injective on the torsion-free arithmetic subgroups and preserves irreducibility and cocompactness as claimed.

Case (3): symplectic groups. Lemma 3.8, case (3) provides algebraic groups $\mathbf{G}$ and $\mathbf{G}'$ that are isomorphic at every finite place, with absolute type C_d. Since k is not totally imaginary and the groups are assumed to have archimedean rank at least 2, Kammeyer and Kionke's congruence subgroup theorem [KK23, Theorem 2.4] applies to these groups of type C_d. Lemmas 3.4 therefore give torsion-free arithmetic subgroups with isomorphic profinite completions.

Since a and b are negative at every archimedean place v, B_v is non-split and isomorphic to $\mathbb{H}$. The factor groups are $\mathrm{Sp}(m,n)$ or the compact form $\mathrm{Sp}(d)$, depending on the sign of $\sigma_v(\alpha)$. Thus

$$\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) \cong \mathrm{Sp}(d)^{r_+} \times \mathrm{Sp}(m,n),$$

and similarly for $\mathbf{G}'$. Projection away from the compact factors is injective on the torsion-free arithmetic lattices and preserves irreducibility and cocompactness, so the conclusion follows. □

3.2.2. *Tits Constructions.* We obtain the following pairs in the exceptional Lie algebra types:

Proposition 3.10. *There exist torsion-free arithmetic lattices $\Gamma < G$, $\Gamma' < G'$ with isomorphic profinite completions if the simple Lie groups G, G' satisfy any of the following:*

- $G = E_{6(2)}$ *and* $G' = E_{6(-14)}$,
- $G = E_{7(7)}$ *and* $G' = E_{7(-25)}$.
- $G = E_{8(8)}$ *and* $G' = E_{8(-24)}$.

Here, Γ and Γ' are cocompact for the E_7 and E_8 cases.

In addition, assuming the congruence subgroup property for anisotropic groups of type E_6, which remains open in general, we can also let $G = E_{6(6)}$ and $G' = E_{6(-26)}$; moreover, Γ and Γ' for both E_6 cases may be chosen cocompact.

For the exceptional real forms, we use Tits' construction from composition algebras and Albert algebras [Tit66a].

Lemma 3.11. *In each of the following cases, there exists a real quadratic field $k = \mathbb{Q}(\alpha)$ with real places v_1, v_2, and two Albert algebras A, A' over k, with the predicted real place identifications:*

- *A_{v_1} is isomorphic to the split real Albert algebra, while A'_{v_1}, A_{v_2} and A'_{v_2} are all isomorphic to the compact one.*
- *A_{v_1} is isomorphic to the split real Albert algebra, A'_{v_1} is isomorphic to the intermediate one, while both A_{v_2} and A'_{v_2} are isomorphic to the compact one.*

Proof. In this proof, we suppose that the two real places are such that $\sigma_{v_1}(\alpha) > 0$ and $\sigma_{v_2}(\alpha) < 0$. Take units $a, b, c' \in k$ that are negative at both real places, and a distinguished unit c with $\sigma_{v_1}(c) > 0$ and $\sigma_{v_2} < 0$; for example $a = b = c' = -1$ and $c = \alpha$. Define the octonion algebras $C = (a, b, c)_k$ and $C' = (a, b, c')_k$, then C and C' satisfies that

$$C_{v_1} \cong \mathbb{O}_s, \quad C'_{v_1} \cong C_{v_2} \cong C'_{v_2} \cong \mathbb{O}.$$

(1) Consider the octonionic Hermitian form $h = \langle 1^{[3]} \rangle$, and Albert algebras

$$A = H_3(C, h), \quad A' = H_3(C', h).$$

Therefore,

$$A_{v_1} \cong H_3(\mathbb{O}_s), \quad A'_{v_1} \cong A_{v_2} \cong A'_{v_2} \cong H_3(\mathbb{O}),$$

which are split and compact, respectively.

(2) Consider the octonionic Hermitian form $h = \langle 1^{[2]}, -\alpha^{[1]} \rangle$, and Albert algebras

$$A = H_3(C, h), \quad A' = H_3(C', h).$$

The construction of h satisfies that

$$h_{v_1} \cong \langle 1^{[2]}, -1^{[1]} \rangle, \quad h_{v_2} \cong \langle 1^{[3]} \rangle.$$

Therefore,

$$A_{v_1} \cong H_3(\mathbb{O}_s), \quad A'_{v_1} \cong H_3(\mathbb{O}, (1, 1, -1)), \quad A_{v_2} \cong A'_{v_2} \cong H_3(\mathbb{O}),$$

which are, respectively, the split, intermediate, and compact real Albert algebras, as desired. □

Remark 3.12. *When $k = \mathbb{Q}$, the construction $C/\mathbb{Q} = (-1, -1, 1)_{\mathbb{Q}}$, $C'/\mathbb{Q} = (-1, -1, -1)_{\mathbb{Q}}$, and $h = \langle 1^{[2]}, -1^{[1]} \rangle$ produces Albert algebras $A/\mathbb{Q}$ and $A'/\mathbb{Q}$ similar to case (2) at the real place, namely, A_∞ is split while A'_∞ is intermediate.*

Proof of Proposition 3.10. Take units $a, b, c \in k$ that satisfy the condition in the proof of Lemma 3.11, e.g., $a = b = -1$ and $c = \alpha$.

Case (1): G is split. Take the quadratic field extension $E = k(\sqrt{c})$, quaternion algebra $B = (b, c)_k$, octonion algebra $C = (a, b, c)_k$, and Albert algebras A, A' in Lemma 3.11, case (1). For the E_6 case, we let

$$\mathbf{G} = G(A, E), \quad \mathbf{G}' = G(A', E).$$

For the E_7 case, we let

$$\mathbf{G} = G(A, B), \quad \mathbf{G}' = G(A', B).$$

For the E_8 case, we let

$$\mathbf{G} = G(A, C), \quad \mathbf{G}' = G(A', C).$$

At each finite place w of k, there is a single isomorphism type of Albert algebra over k_w (see, e.g., [Sch11]), thus $\mathbf{G}_w \cong \mathbf{G}'_w$. On the real places, we have

$$E_{v_1} \cong \mathbb{R} \times \mathbb{R}, \quad E_{v_2} \cong \mathbb{C}, \quad B_{v_1} \cong M_2(\mathbb{R}), \quad B_{v_2} \cong \mathbb{H}, \quad C_{v_1} \cong \mathbb{O}_s, \quad C_{v_2} \cong \mathbb{O}.$$

Using the identification of the real forms arising from Tits's magic-square construction in [BS03, Sections 3 and 7], in the three cases of $\mathbf{G}$ and $\mathbf{G}'$, we have

$$\mathbf{G}_{v_1} \cong E_{6(6)}, \quad \mathbf{G}'_{v_1} \cong E_{6(-26)}, \quad \mathbf{G}_{v_2} \cong \mathbf{G}'_{v_2} \cong E_{6(-78)},$$

or

$$\mathbf{G}_{v_1} \cong E_{7(7)}, \quad \mathbf{G}'_{v_1} \cong E_{7(-25)}, \quad \mathbf{G}_{v_2} \cong \mathbf{G}'_{v_2} \cong E_{7(-133)},$$

or

$$\mathbf{G}_{v_1} \cong E_{8(8)}, \quad \mathbf{G}'_{v_1} \cong E_{8(-24)}, \quad \mathbf{G}_{v_2} \cong \mathbf{G}'_{v_2} \cong E_{8(-248)}.$$

For the E_7 and E_8 groups, the congruence subgroup property is known, so Lemma 3.4 and Corollary 3.5 give torsion-free cocompact lattices with isomorphic profinite completions. The same conclusion holds for the E_6 pair under the stated assumption on the congruence subgroup property for anisotropic groups of type E_6.

Case (2): G is quasi-split. Let A and A' be Albert algebras as in Lemma 3.11, case (2), and quadratic extension $E = k(\sqrt{a})$. Tits' construction gives simply connected groups, [GP07]

$$\mathbf{G} = G(A, E), \quad \mathbf{G}' = G(A', E).$$

Again, $H_3(C)$ and $H_3(C')$ are isomorphic at each finite place w, thus $\mathbf{G}_w \cong \mathbf{G}'_w$. For the two real places v_1 and v_2, $E_{v_1} \cong E_{v_2} \cong \mathbb{C}$ are fields, thus [GP07, 4.4] implies that $\mathbf{G}_{v_i}$ and $\mathbf{G}'_{v_i}$ are of type ${}^2E_6$. In particular, together with the real forms of A and A' as described in Lemma 3.11, the corresponding real forms are identified in [Jac71]:

$$\mathbf{G}_{v_1} \cong E_{6(2)}, \quad \mathbf{G}'_{v_1} \cong E_{6(-14)}, \quad \mathbf{G}_{v_2} \cong \mathbf{G}'_{v_2} \cong E_{6(-78)}.$$

That is to say,

$$\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) \cong E_{6(2)} \times E_{6(-78)}, \quad \mathbf{G}'(k \otimes_{\mathbb{Q}} \mathbb{R}) \cong E_{6(-14)} \times E_{6(-78)}.$$

The compact factor $E_{6(-78)}$ implies that $\mathbf{G}$ and $\mathbf{G}'$ are anisotropic over k. Assuming the congruence subgroup property for these anisotropic k-groups of type ${}^2E_6$, Lemma 3.4 together with Corollary 3.5 thus yields cocompact torsion-free lattices $\Gamma < E_{6(2)}$ and $\Gamma' < E_{6(-14)}$ with isomorphic profinite completions.

Alternatively, consider $k = \mathbb{Q}$, $E = \mathbb{Q}(\sqrt{-1})$, and the Albert algebras $A/\mathbb{Q}$ and $A'/\mathbb{Q}$ resulting from Remark 3.12. Since the Hermitian form $h = \langle 1^{[2]}, -1^{[1]} \rangle$ satisfies the condition in [GP07, Proposition 6.1], the proposition implies that $\mathbb{Q} \times E$ is a subalgebra of both A and A'. By [GP07, Theorem 0.2], $\mathbf{G} = G(A, E)$ and $\mathbf{G}' = G(A', E)$ are isotropic. Therefore, by [KK23, Theorem 2.4], both satisfy the congruence subgroup property. Lemma 3.4 and Corollary 3.5 then yield profinitely isomorphic torsion-free non-cocompact lattices $\Gamma < E_{6(2)}$ and $\Gamma' < E_{6(-14)}$. □

3.2.3. *Outer Real Forms of Type A_{4n-1}.* The remaining type A pairs arise from unitary groups over algebras with involutions of the second kind:

Proposition 3.13. *For every $n \geq 2$, there exist torsion-free cocompact arithmetic lattices*

$$\Gamma < \mathrm{SL}(4n, \mathbb{R}), \qquad \Gamma' < \mathrm{SL}(2n, \mathbb{H}),$$

with $\widehat{\Gamma} \cong \widehat{\Gamma}'$.

Lemma 3.14. *Suppose additionally that α has degree 2, and the minimal polynomial of α is irreducible modulo 2. Let $E = k(\sqrt{a})$, where $a \in \mathcal{O}_k^\times$ such that $\sigma_{v_1}(a) > 0$ and $\sigma_{v_2}(a) < 0$, e.g., $a = \alpha$. Then the central simple E-algebras,*

$$A = M_2(E), \qquad A' = (-1,-1)_E,$$

at the two real places v_1, v_2 of k are

$$A_{v_1} \cong M_2(\mathbb{R}) \times M_2(\mathbb{R}), \qquad A'_{v_1} \cong \mathbb{H} \times \mathbb{H}, \qquad A_{v_2} \cong A'_{v_2} \cong M_2(\mathbb{C}),$$

while $A_w \cong A'_w$ for every finite place w of k.

Proof. The real places of E are

$$E_{v_1} \cong \mathbb{R} \times \mathbb{R}, \qquad E_{v_2} \cong \mathbb{C}.$$

Hence the matrix and quaternionic E-algebras A and A' at the places v_1 and v_2 are

$$A_{v_1} \cong M_2(\mathbb{R}) \times M_2(\mathbb{R}), \qquad A'_{v_1} \cong \mathbb{H} \times \mathbb{H},$$

and

$$A_{v_2} \cong A'_{v_2} \cong M_2(\mathbb{C}).$$

Let w be a finite place of k. If w lies above an odd prime p, then the Hamilton quaternion algebra $(-1,-1)_{\mathbb{Q}}$ splits over $\mathbb{Q}_p$, so

$$A'_w \cong M_2(E_w) \cong A_w.$$

Above 2, irreducibility of the minimal polynomial modulo 2 gives a unique place w with $[k_w : \mathbb{Q}_2] = 2$. The local invariant of $(-1,-1)_{\mathbb{Q}}$ at 2 is $1/2$, hence after restriction to k_w it becomes

$$[k_w : \mathbb{Q}_2] \cdot \frac{1}{2} = 0 \in \mathbb{Q}/\mathbb{Z}.$$

Thus $(-1,-1)_{k_w}$ is split, and again $A'_w \cong M_2(E_w) \cong A_w$. □

Proof of Proposition 3.13. Let k, E, A, A' be as in Lemma 3.14. Let $^-$ denote the nontrivial k-automorphism of E. Define unitary involutions

$$\tau : A \longrightarrow A, \qquad \tau(X) = \overline{X}^{\mathsf{T}},$$

and

$$\tau' : A' \longrightarrow A', \qquad \tau'(x_0 + x_1\mathrm{i} + x_2\mathrm{j} + x_3\mathrm{ij}) = \overline{x_0} - \overline{x_1}\mathrm{i} - \overline{x_2}\mathrm{j} - \overline{x_3}\mathrm{ij}.$$

Using the standard Hermitian form $h = \langle 1^{[2n]} \rangle$, put

$$\mathbf{G} = \mathrm{SU}(2n, A, \tau), \qquad \mathbf{G}' = \mathrm{SU}(2n, A', \tau').$$

At v_1, the involutions identify with

$$\tau : (X, Y) \longmapsto (Y^{\mathsf{T}}, X^{\mathsf{T}}), \qquad \tau' : (x, y) \longmapsto (y^*, x^*),$$

and therefore

$$\mathbf{G}(k_{v_1}) \cong \mathrm{SL}(4n, \mathbb{R}), \qquad \mathbf{G}'(k_{v_1}) \cong \mathrm{SL}(2n, \mathbb{H}).$$

At v_2, both involutions become conjugate transpose on $M_2(\mathbb{C})$, so

$$\mathbf{G}(k_{v_2}) \cong \mathbf{G}'(k_{v_2}) \cong \mathrm{SU}(4n),$$

which is compact.

Let w be a finite place. If E/k splits at w, then

$$A_w \cong A'_w \cong M_2(k_w) \times M_2(k_w),$$

and both local groups are isomorphic to $\mathrm{SL}(4n, k_w)$. If E_w/k_w is a field extension, Morita equivalence identifies

$$\mathrm{SU}(2n, A_w, \tau_w) \cong \mathrm{SU}(4n, E_w/k_w, h_w),$$

and

$$\mathrm{SU}(2n, A'_w, \tau'_w) \cong \mathrm{SU}(4n, E_w/k_w, h'_w),$$

where $h_w \cong \langle 1^{[4n]} \rangle$ and $h'_w \cong h_0^{\perp 2n}$ for a two-dimensional Hermitian form h_0. Hence

$$[\det(h'_w)] = [\det(h_0)^{2n}] = [1] = [\det(h_w)] \quad \text{in} \quad k_w^\times / N_{E_w/k_w}(E_w^\times).$$

The local Hermitian forms therefore have the same determinant invariant, and the corresponding special unitary groups are k_w-isomorphic. Thus $\mathbf{G}$ and $\mathbf{G}'$ are isomorphic at every finite place.

Via Morita equivalence, $\mathbf{G}$ is a special unitary group attached to a $4n$-dimensional Hermitian form over E/k, the congruence subgroup property follows from [KK23]. For $\mathbf{G}'$, the group is a special unitary group of Hermitian dimension $2n$ over the quaternion division algebra $A' = (-1,-1)_E$, with respect to the involution τ' of the second kind. Since it has Hermitian dimension $2n \geq 4$ and real rank $2n-1 > 2$, Tomanov's congruence subgroup theorem for such quaternionic unitary groups applies; see [Rap99, Theorem 3 and the discussion in page 178]. Lemma 3.4 and Corollary 3.5, together with the compact factor at v_2, now give torsion-free cocompact lattices

$$\Gamma < \mathrm{SL}(4n, \mathbb{R}), \qquad \Gamma' < \mathrm{SL}(2n, \mathbb{H}),$$

with isomorphic profinite completions. □

3.3. **Classification of Profinite Rigidity across Real Forms.** Let G and G' be non-isomorphic simply connected simple real Lie groups of real rank at least 2 and of the same absolute Dynkin type. We now classify when they can contain torsion-free lattices with isomorphic profinite completions.

Theorem 3.15. *Let G and G' be non-isomorphic simple real Lie groups of the same absolute Dynkin type. Also, suppose they are simply connected, with real ranks at least 2.*

(1) *If, up to interchanging G and G', the pair falls into one of the following families, then there exist torsion-free lattices $\Gamma < G$ and $\Gamma' < G'$ such that $\widehat{\Gamma} \cong \widehat{\Gamma}'$:*
   - *$G = \mathrm{Spin}(m,n)$ and $G' = \mathrm{Spin}(m',n')$, where $m+n = m'+n' \geq 8$ and*
   $$mn \equiv m'n' \pmod 4;$$
   - *$G = \mathrm{SU}(m,n)$ and $G' = \mathrm{SU}(m',n')$, where $m+n = m'+n' \geq 7$ and*
   $$mn \equiv m'n' \pmod 2;$$
   - *$G = \mathrm{Sp}(m,n)$ and $G' = \mathrm{Sp}(m',n')$, where $m+n = m'+n' \geq 6$;*
   - *$G = \mathrm{SL}(4n, \mathbb{R})$ and $G' = \mathrm{SL}(2n, \mathbb{H})$, where $n \geq 2$;*
   - *$(G, G') = (E_{6(-14)}, E_{6(2)})$, $(E_{7(-25)}, E_{7(7)})$, or $(E_{8(-24)}, E_{8(8)})$.*

(2) *Assuming the congruence subgroup property for anisotropic groups of type E_6, the additional pair $(E_{6(-26)}, E_{6(6)})$ also satisfies (1).*

(3) *In all other cases, lattices $\Gamma < G$ and $\Gamma' < G'$ have non-isomorphic profinite completions.*

Remark 3.16. *The* Spin, SU *or* SL, *and* Sp *groups occurring in Theorem 3.15 have absolute Dynkin types B_ℓ or D_ℓ, A_ℓ, and C_ℓ, respectively. The noncompact real forms of classical Lie algebras are summarized in Table 1 in Appendix A.*

Before proving the theorem, we exclude two families using their Brauer classes.

Lemma 3.17. *Suppose either*

$$G \cong \mathrm{Sp}(m,n), \qquad G' \cong \mathrm{Sp}(2(m+n), \mathbb{R}),$$

or

$$G \cong \mathrm{Spin}(m,n), \qquad G' \cong \mathrm{Spin}^*(m+n),$$

where in the second case $m+n$ is even and at least 10. Assume that both groups have real rank at least 2. Then for any lattices $\Gamma < G$ and $\Gamma' < G'$,

$$\widehat{\Gamma} \ncong \widehat{\Gamma}'.$$

Proof. Suppose to the opposite that $\widehat{\Gamma} \cong \widehat{\Gamma}'$. By Margulis arithmeticity, after replacing Γ, Γ' by corresponding finite-index subgroups, we may assume that they are arithmetic subgroups of simply connected absolutely almost simple groups $\mathbf{G}/k$ and $\mathbf{G}'/l$, respectively, where k and l are totally real. There are distinguished real places v of k and v' of l such that

$$\mathbf{G}(k_v) \cong G, \qquad \mathbf{G}'(l_{v'}) \cong G',$$

while the groups at all other real places are compact. By Theorem 2.10, the finite places of k and l are matched so that the corresponding local fields and algebraic groups are isomorphic. The adeles isomorphism also implies the degrees $[k : \mathbb{Q}] = [l : \mathbb{Q}]$.

In the symplectic case, let $\mathbf{G}_0$ be the split simply connected group of the corresponding type C_d. Following [EK24, Proposition 11], the k- and l-forms $\mathbf{G}$ and $\mathbf{G}'$ determine classes

$$\alpha \in H^1(k, \operatorname{Ad}\mathbf{G}_0), \qquad \beta \in H^1(l, \operatorname{Ad}\mathbf{G}_0).$$

For $F = k$ or l, the Kummer sequence $1 \to \mu_2 \to \mathbf{G}_0 \to \operatorname{Ad}\mathbf{G}_0 \to 1$ yields a boundary map $\delta_F : H^1(F, \operatorname{Ad}\mathbf{G}_0) \to H^2(F, \mu_2)$ to Tits classes, which are regarded as Brauer classes via the Kummer identification $H^2(F, \mu_2) \cong \operatorname{Br}(F)[2]$. At corresponding finite places, the localizations of $\delta_k(\alpha)$ and $\delta_l(\beta)$ agree by adelic superrigidity. Since $k/\mathbb{Q}$ and $l/\mathbb{Q}$ are totally real and have the same degree, they have exactly the same number of nondistinguished real places, where both groups are the compact real form $\mathrm{Sp}(d)$, so the corresponding local classes agree there as well. The Albert–Brauer–Hasse–Noether reciprocity law therefore forces

$$\delta_{\mathbb{R}}(\alpha_v) = \delta_{\mathbb{R}}(\beta_{v'}) \in \operatorname{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$$

at the distinguished real place. However, the split form $\mathrm{Sp}(2d, \mathbb{R})$ has trivial class in $\operatorname{Br}(\mathbb{R})$, whereas every noncompact quaternionic form $\mathrm{Sp}(m, n)$ has the nontrivial class. This is a contradiction.

For the spin case, following [EK24, Proposition 14], realize $\mathbf{G}$ and $\mathbf{G}'$ by central simple algebras A/k and A'/l equipped with involutions of the first kind and orthogonal type. At corresponding finite places, the local isomorphisms of the algebraic groups induce isomorphisms of the associated Lie algebras. By [Jac79, Chapter X, Theorem 12], an isomorphism between the Lie algebra of skew elements associated with central simple algebras with involution extends to an isomorphism of the underlying algebras that identifies the involutions. Hence, at corresponding finite places, their local Brauer classes agree. The same is true at every nondistinguished real place, where both groups are compact. Hence, the Albert–Brauer–Hasse–Noether reciprocity law gives

$$[A_v] = [A'_{v'}] \in \operatorname{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$$

at the distinguished real place. However, for a real orthogonal form $\mathrm{Spin}(m, n)$ the underlying central simple algebra is split, so $[A_v] = 0$, whereas $\mathrm{Spin}^*(2d)$ corresponds to the quaternionic algebra $M_d(\mathbb{H})$ and therefore has the nontrivial class. This is again a contradiction. □

Proof of Theorem 3.15. The existence statements in (1) are supplied by Proposition 3.6, Proposition 3.13, and Proposition 3.10, which in particular cover the only E_8 pair. The conditional E_6 case is exactly (2).

For (3), suppose that lattices $\Gamma < G$ and $\Gamma' < G'$ satisfy $\widehat{\Gamma} \cong \widehat{\Gamma}'$. No such pair occurs in types F_4 or G_2. Indeed, type G_2 has a unique noncompact real form, namely $G_{2(2)}$. Type F_4 has two noncompact real forms, $F_{4(-20)}$ and $F_{4(4)}$, but the former has real rank 1. Thus neither type admits two non-isomorphic real forms both of real rank at least 2.

By Proposition 2.13, G and G' must either both be inner forms or both be outer forms. This excludes the inner–outer pairs of types A_ℓ, D_ℓ, and E_6. The relevant classical pairs are listed in Table 1, while the excluded E_6 pairs are precisely those with one member in $\{E_{6(-26)}, E_{6(6)}\}$ and the other in $\{E_{6(-14)}, E_{6(2)}\}$.

Applying restriction of scalars to $\mathbb{Q}$, Theorem 2.10 and Proposition 2.11 imply that the dimensions of the associated symmetric spaces are congruent modulo 4. For the classical real forms, the dimensions used below are recorded in Table 1. This excludes the following remaining cases:

- For $G \cong \mathrm{Spin}(m,n)$ and $G' \cong \mathrm{Spin}(m',n')$ with $d = m+n = m'+n'$ odd, both groups have absolute Dynkin type $B_{(d-1)/2}$. We exclude the case in which one of m,n is $0 \pmod 4$ and one of m',n' is $2 \pmod 4$, because
$$\dim X = mn \equiv 0 \pmod 4, \qquad \dim X' = m'n' \equiv 2 \pmod 4.$$
- For $G \cong \mathrm{SU}(m,n)$ and $G' \cong \mathrm{SU}(m',n')$ with $d = m+n = m'+n'$, both groups have absolute Dynkin type A_{d-1}. We exclude the case in which m,n are both even and m',n' are both odd, because
$$\dim X = 2mn \equiv 0 \pmod 4, \qquad \dim X' = 2m'n' \equiv 2 \pmod 4.$$
- For $G \cong \mathrm{SL}(2n,\mathbb{R})$ and $G' \cong \mathrm{SL}(n,\mathbb{H})$, both groups have absolute Dynkin type A_{2n-1}. We exclude the case in which n is odd, because
$$\dim X = (2n-1)(n+1), \qquad \dim X' = (2n+1)(n-1),$$
and $\dim X - \dim X' = 2n \equiv 2 \pmod 4$.
- For the exceptional type E_7, we exclude the pairs involving $E_{7(-5)}$ and either $E_{7(-25)}$ or $E_{7(7)}$, since their symmetric-space dimensions are respectively 64, 54, and 70.

After these exclusions, the only remaining pairs are exactly the two families treated in Lemma 3.17: the pair $\mathrm{Sp}(m,n)$ and $\mathrm{Sp}(2d,\mathbb{R})$ of type C_d, and the pair $\mathrm{Spin}(m,n)$ and $\mathrm{Spin}^*(d)$ of type $D_{d/2}$. The constructions for (1), (2) in the preceding subsections along with these exclusions for (3) exhaust all pairs of real forms. □

4. Profinitely Isomorphic Lattices in Semisimple Lie Groups

Extending the classical constructions in Section 3, we now allow several noncompact archimedean factors, including split real and complex factors. The resulting lattices are irreducible in the ambient products. We also discuss when these constructions can be made explicit. Both the existence results and the explicit constructions are needed for the proof of Theorem A.

4.1. Constructions in Semisimple Lie Groups.

The following theorem establishes the existence of profinitely isomorphic lattices in pairs of semisimple Lie groups whose factors include different noncompact spin, unitary, or quaternionic symplectic groups, together with possible real split factors.

Theorem 4.1. *Let $r_-, r_1, r_2, r_s \in \mathbb{Z}_{\geq 0}$ with $r_1 + r_2 = r_-$ and $r_2 \geq 1$, and let $m_1, n_1, m_2, n_2 \geq 1$ satisfy $m_1 + n_1 = m_2 + n_2 = d$. In each of the following cases, let G and G' be the displayed Lie groups and assume that $\mathrm{rk}_{\mathbb{R}} G, \mathrm{rk}_{\mathbb{R}} G' \geq 2$:*

(1) *The quadratic form has dimension $d \geq 5$,*
$$G = \mathrm{Spin}(m_1,n_1)^{r_-}, \quad G' = \mathrm{Spin}(m_1,n_1)^{r_1} \times \mathrm{Spin}(m_2,n_2)^{r_2},$$
and one of the following holds:
- *$4 \mid (m_1 - m_2)$ or $4 \mid (m_1 - n_2)$;*
- *$2 \mid (m_1 - m_2)$ or $2 \mid (m_1 - n_2)$, and $2 \mid r_2$.*

(2) *The Hermitian form has dimension $d \geq 3$,*
$$G = \mathrm{SU}(m_1,n_1)^{r_-} \times \mathrm{SL}(d,\mathbb{R})^{r_s}, \quad G' = \mathrm{SU}(m_1,n_1)^{r_1} \times \mathrm{SU}(m_2,n_2)^{r_2} \times \mathrm{SL}(d,\mathbb{R})^{r_s},$$
and either $2 \mid (m_1 - m_2)$ or $2 \mid (m_1 - n_2)$.

(3) *The Hermitian form has dimension $d \geq 2$, and*
$$G = \mathrm{Sp}(m_1,n_1)^{r_-} \times \mathrm{Sp}(2d,\mathbb{R})^{r_s}, \quad G' = \mathrm{Sp}(m_1,n_1)^{r_1} \times \mathrm{Sp}(m_2,n_2)^{r_2} \times \mathrm{Sp}(2d,\mathbb{R})^{r_s}.$$

Then there exist irreducible torsion-free cocompact arithmetic lattices $\Gamma < G$ and $\Gamma' < G'$ with $\widehat{\Gamma} \cong \widehat{\Gamma}'$.

Remark 4.2. *After our first draft, Py and Isenrich pointed us to Aka's work. In* [Aka12, Theorem 1]*, Aka also constructed pairs of simply-connected absolutely almost simple algebraic groups of the same classical absolute Dynkin type ($D_{(n+1)/2}$ if n is odd and $B_{n/2}$ if n is even) in $\mathrm{Spin}(1,n)$ and $\mathrm{Spin}(5,n-4)$ for $n \geq 6$. Note that the specific setting in* [Aka12] *differs from*

ours (from the number field to the ambient groups), and we can't see how Aka's construction applies to deduce a result similar to Theorem A. Also, after our first draft, Holger Kammeyer and Adrian Baumann pointed us to their recent work [BK26]*, where they classified "congruence rigid" algebraic groups. Their result is related to Theorem B but with different techniques. Moreover, Theorem B with extended results in Section 4 focus on explicit constructions.*

The finite-place calculations in Lemma 3.8 apply to the semisimple case with only an additional $2 \mid (m_1 - m_2)$ and even r_2 treatment in the spin case:

Lemma 4.3. *Let both α and $\alpha - 1$ be algebraic units, such that α is real at every archimedean place, put $k = \mathbb{Q}(\alpha)$, and assume $m_1 > m_2$.*

- *Suppose that k has r_+ real places at which $\alpha > 1$, r_1 real places at which $0 < \alpha < 1$, and r_2 real places at which $\alpha < 0$, thus $[k : \mathbb{Q}] = r_+ + r_1 + r_2$. Suppose that $r_+ \geq 1$ and that $r_1, r_2, m_1, n_1, m_2, n_2$ and d satisfy the hypotheses of Theorem 4.1(1), with the divisibility conditions restricted to those involving $(m_1 - m_2)$. In the second alternative of that theorem, assume in addition that the ideal $(2) \subset \mathcal{O}_k$ is prime. Define*

$$q = \langle 1^{[m_1]}, (\alpha - 1)^{[n_1]} \rangle, \qquad q' = \langle 1^{[m_2]}, \alpha^{[m_1 - m_2]}, (\alpha - 1)^{[n_1]} \rangle.$$

  Let $\mathbf{G} = \mathrm{Spin}(q)$ and $\mathbf{G}' = \mathrm{Spin}(q')$ be the associated spin k-groups. Then

$$\mathbf{G} \times_k k_v \cong \mathbf{G}' \times_k k_v$$

  for every finite place v of k.
- *Let $a \in \mathcal{O}_k^\times$, and let $E = k(\sqrt{a})$ be the corresponding quadratic field extension. Suppose that there are r_s real places with $\sigma_v(a) > 0$; among the other real places where $\sigma_v(a) < 0$, there are r_+ places at which $\alpha > 1$, r_1 places at which $0 < \alpha < 1$, and r_2 places at which $\alpha < 0$. Assume $r_+ \geq 1$ and that the remaining parameters satisfy Theorem 4.1(2), with the divisibility conditions restricted to those involving $(m_1 - m_2)$. Define the E/k-Hermitian forms*

$$h = \langle 1^{[m_1]}, (\alpha - 1)^{[n_1]} \rangle, \qquad h' = \langle 1^{[m_2]}, \alpha^{[m_1 - m_2]}, (\alpha - 1)^{[n_1]} \rangle,$$

  and let $\mathbf{G} = \mathrm{SU}(h, E/k)$ and $\mathbf{G}' = \mathrm{SU}(h', E/k)$. Then

$$\mathbf{G} \times_k k_v \cong \mathbf{G}' \times_k k_v$$

  for every finite place v of k.
- *Let $a, b \in \mathcal{O}_k^\times$, and $B = (a, b)_k$ be the corresponding quaternion algebra. Suppose that there are r_s real places at which B splits; among the other real places where B is ramified, there are r_+ places at which $\alpha > 1$, r_1 places at which $0 < \alpha < 1$, and r_2 places at which $\alpha < 0$. Assume $r_+ \geq 1$ and that the remaining parameters satisfy Theorem 4.1(3). Define Hermitian forms with respect to the canonical involution on B by*

$$h = \langle 1^{[m_1]}, (\alpha - 1)^{[n_1]} \rangle, \qquad h' = \langle 1^{[m_2]}, \alpha^{[m_1 - m_2]}, (\alpha - 1)^{[n_1]} \rangle,$$

  and let $\mathbf{G} = \mathrm{Sp}(h, B/k)$ and $\mathbf{G}' = \mathrm{Sp}(h', B/k)$. Then

$$\mathbf{G} \times_k k_v \cong \mathbf{G}' \times_k k_v$$

  for every finite place v of k.

Proof. The proof is the same as that of Lemma 3.8: the common $(\alpha - 1)$-block does not affect the determinant comparison, and in the spin case, the ratio of the Hasse invariants is

$$\frac{\epsilon_v(q')}{\epsilon_v(q)} = (\alpha, \alpha)_v^{(m_1 - m_2)(m_1 - m_2 - 1)/2} (\alpha, \alpha - 1)_v^{(m_1 - m_2) n_1} = (\alpha, \alpha)_v^{(m_1 - m_2)(m_1 - m_2 - 1)/2}.$$

In the first alternative of the spin case we deduce that $\epsilon_v(q') = \epsilon_v(q)$; in the second alternative, i.e., when $2 \mid (m_1 - m_2)$ and $2 \mid r_2$, it remains to show that $(\alpha, \alpha)_v = 1$ at every finite place. If v lies above an odd rational prime, then $\alpha^{-1} \in \mathcal{O}_{k_v}^\times$ is a sum of two squares in k_v; equivalently,

$$(\alpha, \alpha)_v = 1.$$

At a real place, $(\alpha,\alpha)_v = -1$ precisely when α is negative. There are r_2 such places, which is an even number; thus, the product of the archimedean symbols is 1. Finally, because (2) is prime in $\mathcal{O}_k$, there is a unique place v_2 above 2. Hilbert reciprocity therefore gives

$$(\alpha,\alpha)_{v_2} = 1.$$

Thus $(\alpha,\alpha)_v = 1$ at every finite place, so $\epsilon_v(q) = \epsilon_v(q')$, and the k_v-isomorphism follows. □

Proposition 4.4. *In the setting of Lemma 4.3:*

- *In the spin case, there are natural homomorphisms*

$$\mathbf{G}(k) \longrightarrow \mathrm{Spin}(m_1,n_1)^{r_-}, \qquad \mathbf{G}'(k) \longrightarrow \mathrm{Spin}(m_1,n_1)^{r_1} \times \mathrm{Spin}(m_2,n_2)^{r_2},$$

  obtained from the diagonal archimedean embedding followed by projection away from the compact factors. Moreover, there exist irreducible torsion-free cocompact arithmetic lattices in these two Lie groups whose profinite completions are isomorphic.
- *In the unitary settings, there are natural homomorphisms*

$$\mathbf{G}(k) \longrightarrow \mathrm{SU}(m_1,n_1)^{r_-} \times \mathrm{SL}(d,\mathbb{R})^{r_s},$$

$$\mathbf{G}'(k) \longrightarrow \mathrm{SU}(m_1,n_1)^{r_1} \times \mathrm{SU}(m_2,n_2)^{r_2} \times \mathrm{SL}(d,\mathbb{R})^{r_s},$$

  and irreducible torsion-free cocompact arithmetic lattices in these two Lie groups whose profinite completions are isomorphic.
- *In the symplectic settings, there are natural homomorphisms*

$$\mathbf{G}(k) \longrightarrow \mathrm{Sp}(m_1,n_1)^{r_-} \times \mathrm{Sp}(2d,\mathbb{R})^{r_s},$$

$$\mathbf{G}'(k) \longrightarrow \mathrm{Sp}(m_1,n_1)^{r_1} \times \mathrm{Sp}(m_2,n_2)^{r_2} \times \mathrm{Sp}(2d,\mathbb{R})^{r_s},$$

  and irreducible torsion-free cocompact arithmetic lattices in these two Lie groups whose profinite completions are isomorphic.

Proof. The groups $\mathbf{G}$ and $\mathbf{G}'$ are simply connected and absolutely almost simple. Since $r_+ \geq 1$, the field k is not totally imaginary, and the rank assumptions give archimedean rank at least 2 for both groups. The same congruence subgroup theorems used in the proof of Proposition 3.6 apply. Together with Lemma 4.3, Lemma 3.4 therefore yields torsion-free arithmetic subgroups with isomorphic profinite completions.

It remains to identify their noncompact archimedean factors. There are potentially archimedean places where $\mathbf{G}$ splits or is complex, and potentially more archimedean places with $\sigma(\alpha) < 0$:

(1) At a real place v with $\sigma_v(\alpha) > 1$, the forms q, q' are positive definite and the corresponding factor is $\mathrm{Spin}(d)$. At a real place with $0 < \sigma_v(\alpha) < 1$, the factor for both is $\mathrm{Spin}(m_1,n_1)$. At a real place with $\sigma_v(\alpha) < 0$, the factor for $\mathbf{G}$ is still $\mathrm{Spin}(m_1,n_1)$, while which for $\mathbf{G}'$ is $\mathrm{Spin}(m_2,n_2)$. Hence

$$\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) \cong \mathrm{Spin}(d)^{r_+} \times \mathrm{Spin}(m_1,n_1)^{r_-},$$

and

$$\mathbf{G}'(k \otimes_{\mathbb{Q}} \mathbb{R}) \cong \mathrm{Spin}(d)^{r_+} \times \mathrm{Spin}(m_1,n_1)^{r_1} \times \mathrm{Spin}(m_2,n_2)^{r_2}.$$

(2) At a real place with $\sigma_v(a) > 0$, the quadratic algebra E_v is split and the factor is $\mathrm{SL}(d,\mathbb{R})$. At a real place with $\sigma_v(a) < 0$, one has $E_v \cong \mathbb{C}$; the factor is $\mathrm{SU}(d)$, $\mathrm{SU}(m_1,n_1)$, or $\mathrm{SU}(m_2,n_2)$ for the corresponding cases similar to (1). Consequently,

$$\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) \cong \mathrm{SU}(d)^{r_+} \times \mathrm{SU}(m_1,n_1)^{r_-} \times \mathrm{SL}(d,\mathbb{R})^{r_s},$$

and $\mathbf{G}'(k \otimes_{\mathbb{Q}} \mathbb{R})$ is isomorphic to a mixed product in a similar way.

(3) At a real place where B splits, Morita equivalence gives the split factor $\mathrm{Sp}(2d,\mathbb{R})$. At a real place where B is ramified, the factor is $\mathrm{Sp}(d)$, $\mathrm{Sp}(m_1,n_1)$, or $\mathrm{Sp}(m_2,n_2)$ for the corresponding cases. Thus

$$\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) \cong \mathrm{Sp}(d)^{r_+} \times \mathrm{Sp}(m,n)^{r_-} \times \mathrm{Sp}(2d,\mathbb{R})^{r_s},$$

and $\mathbf{G}'(k \otimes_{\mathbb{Q}} \mathbb{R})$ is isomorphic to a mixed product in a similar way. Since $r_+ \geq 1$, both groups have a compact archimedean factor. Corollary 3.5 therefore gives the required irreducible torsion-free cocompact lattices in the displayed noncompact factors. □

To complete the proof of Theorem 4.1, it remains to choose the number field with the expected archimedean behavior:

Proof of Theorem 4.1. Since $\mathrm{Spin}(m_2, n_2) \cong \mathrm{Spin}(n_2, m_2)$, we may interchange m_2 and n_2 and assume that $4 \mid (m_1 - m_2)$ in the first alternative of Theorem 4.1(1), or $2 \mid (m_1 - m_2)$ in the second. Similarly, since $\mathrm{SU}(m_2, n_2) \cong \mathrm{SU}(n_2, m_2)$, we assume $2 \mid (m_1 - m_2)$ in Theorem 4.1(2). Up to exchanging m_1 with n_1 and m_2 with n_2, we assume $m_1 > m_2$. In all three cases, we fix an auxiliary integer $r_+ \geq 2$.

(1) We claim the existence of an algebraic unit α such that $\alpha - 1 \in \mathcal{O}^\times_{\mathbb{Q}(\alpha)}$ and with predicted numbers of real places in the three intervals. Let

$$D = r_- + r_+ = r_1 + r_2 + r_+.$$

Choose a polynomial $g(x) \in \mathbb{Z}[x]$ of degree $D - 3$ with positive leading term denoted by c, $r_+ - 2$ distinct real roots $x_i > 1$, r_1 distinct roots $0 < x_i < 1$, and $r_2 - 1$ distinct negative roots. Choose an irreducible polynomial $\overline{f}(x) \in \mathbb{F}_2[x]$ of degree D, and a monic lift $f_0(x) \in \mathbb{Z}[x]$ with $f_0(1) = (-1)^{r_+}$ and $f_0(0) = (-1)^{r_+ + r_1}$. Set

$$f_N(x) = f_0(x) - 2Nx(x-1)g(x).$$

Then $(2N)^{-1} f_N(x) \to -x(x-1)g(x)$ coefficientwise away from the leading term as $N \to \infty$. For all sufficiently large N, the polynomial f_N has $r_+ - 2$ roots $\alpha_i > 1$, r_1 roots $0 < \alpha_i < 1$, $r_2 - 1$ negative roots, and additional real roots α_0, α_1 and α_∞ sufficiently close to 0, 1, and with sufficiently large absolute value, respectively; all roots are distinct. Since f_N is monic, the sum of the roots is $2Nc + O(1) \to \infty$, so $\alpha_\infty > 0$; the product of the roots is $(-1)^D f_N(0) = (-1)^{r_2}$, and $\prod(\alpha_i - 1) = (-1)^D f_N(1) = (-1)^{r_-}$, so $\alpha_1 > 1$ and $\alpha_0 < 0$. Hence f_N has exactly r_+ roots $\alpha_i > 1$, r_1 roots $0 < \alpha_i < 1$, and r_2 negative roots.

The reduction of f_N modulo 2 is $\overline{f}$, so f_N is irreducible over $\mathbb{Q}$. Let α be a root and put $k = \mathbb{Q}(\alpha)$. The embeddings $k \hookrightarrow \mathbb{C}$ are obtained by sending α to the roots of f_N, so k has precisely the prescribed archimedean sign pattern. Since $\overline{f}$ is irreducible and separable, (2) is prime in $\mathcal{O}_k$. Proposition 4.4, case (1) now gives the required lattices.

(2) We claim the existence of an algebraic unit α, and $a \in \mathcal{O}^\times_{\mathbb{Q}(\alpha)}$, such that $\alpha - 1 \in \mathcal{O}^\times_{\mathbb{Q}(\alpha)}$ with predicted numbers of split places and non-split places in the three intervals. If $r_s = 0$, we construct $f = f_N$ same as above, and take $a = -1$. If $r_s \geq 1$, we modify the construction by setting

$$f_N(x) = f_0(x) - 2Nx(x-1)(x+1)g(x),$$

where the monic lift $f_0 \in \mathbb{Z}[x]$ satisfies $f_0(1) = (-1)^{r_+}$, $f_0(0) = (-1)^{r_+ + r_1}$ and $f_0(-1) = (-1)^{r_+ + r_-}$, and the polynomial $g \in \mathbb{Z}[x]$ of degree $D - 4$ with positive leading term has $r_+ - 2$ distinct roots $x_i > 1$, r_1 distinct roots $0 < x_i < 1$, $r_2 - 1$ distinct roots $-1 < x_i < 0$, and $r_s - 1$ distinct roots $x_i < -1$. As $N \to \infty$, a similar Vieta argument guarantees that $f = f_N$ has exactly r_+, r_1, r_2, and r_s roots in these four intervals. Choosing $a = -\alpha - 1$, the condition $|f(-1)| = 1$ implies that a is a unit, and the number of roots of f in each interval gives the prescribed r_s split real places of $E = k(\sqrt{a})$, and the r_+, r_1 and r_2 ramified places in each cases. Proposition 4.4, case (2) gives the required lattices.

(3) Choose α with the required sign pattern as above, and take $a = b = -1$ or $a = b = -\alpha - 1$ corresponding to case (2), so the quaternion algebra $B = (a, b)_k$ splits at the prescribed r_s real places and is ramified at the prescribed r_+, r_1 and r_2 real places. Proposition 4.4, case (3) then gives the required lattices. □

Remark 4.5. *We can allow complex factors* $\mathrm{Spin}(d, \mathbb{C})$, $\mathrm{SU}(d, \mathbb{C})$, *and* $\mathrm{Sp}(2d, \mathbb{C})$ *in the three cases of Theorem 4.1, respectively, by allowing the field* $\mathbb{Q}(\alpha)$ *to have complex places. This is achieved by including pairs of nonreal conjugate roots in the auxiliary polynomial* $g(x)$.

In the unitary case, the parity restrictions on the signatures can also be replace by the condition of $2 \mid r_2$. *This is achieved by a Hilbert reciprocity argument analogous to the spin case.*

4.2. **Explicit Construction of Profinitely Isomorphic Lattices.** For certain cases of the semisimple classical (Theorem 4.1) and exceptional (Proposition 3.10 constructions, there exists level-3 congruence subgroups $\Gamma < \mathbf{G}$, $\Gamma' < \mathbf{G}'$ with the prescribed profinite isomorphism. The reason is the triviality of the congruence kernels and compatible integral structures at all finite places.

Lemma 4.6. *Let $\mathbf{G}$ be a connected, simply connected, absolutely almost simple algebraic group over a number field k. Assume that $\mathbf{G}$ satisfies the congruence subgroup property, and at some real place v, the Lie group $\mathbf{G}(k_v)$ is isomorphic to one of the following:*

- $\mathrm{Spin}(m,n)$ *with* $m+n \geq 5$ *and* $m,n \geq 2$.
- $\mathrm{SU}(m,n)$ *with* $m+n \geq 3$, *or* $\mathrm{SL}(d,\mathbb{R})$ *with* $d \geq 3$.
- $\mathrm{Sp}(2d,\mathbb{R})$, $d \geq 2$.
- *Exceptional real forms* $E_{7(-25)}$, $E_{7(7)}$, $E_{8(-24)}$, *or* $E_{8(8)}$.

Then the congruence kernel of every arithmetic lattice $\Gamma < \mathbf{G}(k)$ is trivial.

Proof. Since $\mathbf{G}$ is assumed to satisfy the CSP, [PR10, Theorem 2] then implies that this congruence kernel is isomorphic to the Pontrjagin dual of the metaplectic kernel. At the distinguished real place v, $\mathbf{G}(k_v)$ is isomorphic to one of the real forms listed above, which is not topologically simply connected. Therefore, [PR10, Theorem 3] implies that the metaplectic kernel of $\mathbf{G}$ is trivial, and the congruence kernel of $\mathbf{G}$ is also trivial. Consequently, the arithmetic and congruence topologies coincide on $\mathbf{G}(k)$, and hence on every arithmetic subgroup of $\mathbf{G}(k)$. □

Consider the following pairs of Lie groups G, G':

- G, G' are Lie groups in Theorem 4.1, case (1), with $m_1, n_1 \geq 2$, either $m_2, n_2 \geq 2$ or $r_1 > 0$, and either $4 \mid (m_1 - m_2)$ or $2 \mid (m_1 - m_2)$ and $2 \mid r_2$. This includes the corresponding case in Proposition 3.6.
- G, G' are Lie groups in Theorem 4.1, case (2), with $2 \mid (m_1 - m_2)$. This includes the corresponding case in Proposition 3.6.
- G, G' are Lie groups in Theorem 4.1, case (3), with $r_s \geq 1$.
- G, G' are Lie groups in Proposition 3.10 corresponding to the Lie algebras E_7 or E_8.

Let $\mathbf{G}$ and $\mathbf{G}'$ be the algebraic k-groups constructed in the corresponding proofs. For each $\mathbf{G}$, we choose a faithful k-representation $\rho : \mathbf{G} \to \mathrm{GL}_k(W)$ and a full $\mathcal{O}_k$-lattice $\Lambda \subset W$ as follows.

- For $\mathbf{G} = \mathrm{Spin}(k,q)$, take $W = \mathrm{Cl}^+(k^d, q)$ and let ρ be the representation by left multiplication. Set
$$\Lambda = \mathscr{C}^+(k^d, q) = \bigoplus_{\substack{0 \leq 2s \leq d \\ 1 \leq i_1 < \cdots < i_{2s} \leq d}} \mathcal{O}_k e_{i_1} \cdots e_{i_{2s}}.$$
- For $\mathbf{G} = \mathrm{SU}(E/k, h)$, we take the matrix representation on $W = E^d$, seen as a k-vector space, and $\Lambda = \mathcal{O}_E^d$.
- For $\mathbf{G} = \mathrm{Sp}(B/k, h)$, we take the matrix representation on $W = B^d$, seen as a k-vector space, and $\Lambda = \mathcal{O}_B^d$, where $\mathcal{O}_B$ is a fixed maximal $\mathcal{O}_k$-order in B.
- For $\mathbf{G} = G(H_3(C), B)$, use its realization as the isometry group of the generalized Freudenthal triple system on $\mathrm{Mat}_{28}(B)$. Take the natural matrix representation on $W = B^{28}$, viewed as a 112-dimensional k-vector space.
- For $\mathbf{G} = G(H_3(C), C)$, take $W = \mathrm{Lie}(\mathbf{G})$ and $\rho = \mathrm{Ad}$, so that $\dim_k W = 248$.

We define
$$\Gamma = \{g \in \mathbf{G}(k) : \rho(g)\Lambda = \Lambda,\ (\rho(g) - \mathrm{id}_W)\Lambda \subseteq 3\Lambda\},$$
and define $\Gamma' \leq \mathbf{G}'(k)$ similarly. We identify these subgroups with their images in G and G' under the archimedean projections. In the exceptional cases, the full $\mathcal{O}_k$-lattices $\Lambda \subset W$ and $\Lambda' \subset W'$ are chosen so that, at every finite place v, there exist a k_v-isomorphism $\varphi_v : \mathbf{G}_{k_v} \to \mathbf{G}'_{k_v}$ and a linear isomorphism $f_v : W_v \to W'_v$ satisfying
$$f_v \rho(g) = \rho'(\varphi_v(g)) f_v, \quad f_v \Lambda_v = \Lambda'_v. \tag{1}$$
Such choices will be constructed in the proof below.

Lemma 4.7. *Under these settings, we have*

$$\prod_{v\in V_f(k)} \Gamma_v \cong \widehat{\Gamma}.$$

Proof. Since the algebraic group $\mathbf{G}$ is connected, simply connected, absolutely almost simple and has at least one noncompact archimedean place, strong approximation implies that the image of the diagonal embeddings $\mathbf{G}(k) \hookrightarrow \mathbf{G}(\mathbb{A}_k^f)$ is dense [PR, Theorem 7.12]. Moreover, the relations

$$W \cap \prod_{v\in V_f(k)} \Lambda_v = \Lambda, \quad W \cap \prod_{v\in V_f(k)} 3\Lambda_v = 3\Lambda,$$

together with the level-3 definition of Γ, imply the identification

$$\Gamma = \mathbf{G}(k) \cap \prod_{v\in V_f(k)} \Gamma_v.$$

The denseness of $\mathbf{G}(k)$ thus implies that Γ is dense in $\prod_{v\in V_f(k)} \Gamma_v$, so the congruence completion

$$\overline{\Gamma} = \prod_{v\in V_f(k)} \Gamma_v.$$

By Lemma 4.6, both Γ and Γ' have trivial congruence kernels. Therefore,

$$\widehat{\Gamma} \cong \overline{\Gamma} \cong \prod_{v\in V_f(k)} \Gamma_v,$$

□

Theorem 4.8. *Suppose that the minimal polynomial of α over $\mathbb{Q}$ is irreducible modulo 2. Then $\Gamma < G$ and $\Gamma' < G'$ are irreducible, torsion-free, cocompact arithmetic lattices with isomorphic profinite completions.*

Proof. It is clear that level-3 congruence subgroups are torsion-free, and Corollary 3.5 guarantees the irreducibility and cocompactness.

For the profinitely isomorphism, Lemma 4.7 identifies $\widehat{\Gamma} = \prod_{v\in V_f(k)} \Gamma_v$ and $\widehat{\Gamma}' = \prod_{v\in V_f(k)} \Gamma'_v$. Thus, it suffices to identify an appropriate isomorphism between the adelic groups,

$$\Phi : \mathbf{G}(\mathbb{A}_k^f) \xrightarrow{\sim} \mathbf{G}'(\mathbb{A}_k^f),$$

that is restricted to an isomorphism between $\prod_{v\in V_f(k)} \Gamma_v$ and $\prod_{v\in V_f(k)} \Gamma'_v$.

Case (1): Classical groups. For the spin, unitary, and symplectic cases, our construction assumed $m_1 > m_2$ without loss of generality. In the spin case, we claim that at every finite place v, there exists numbers $x_v, y_v, z_v, w_v \in \mathcal{O}_{k,v}$ such that

$$x_v^2 + y_v^2 + z_v^2 + w_v^2 = \alpha.$$

Indeed, for a place v above an odd prime, $[\alpha] \in \mathbb{F}_q$ is a sum of two squares, and the Hensel lift gives a two-square expression for α. For $v \mid 2$, the irreducibility of the minimal polynomial implies $k_v/\mathbb{Q}_2$ is unramified. Therefore, [HH23, Proposition 4.3] guarantees a four-square expression for α.

Now, taking the matrix,

$$R_0 = \begin{pmatrix} x_v & -y_v & -z_v & -w_v \\ y_v & x_v & -w_v & z_v \\ z_v & w_v & x_v & -y_v \\ w_v & -z_v & y_v & x_v \end{pmatrix} \in \mathrm{Mat}_4(\mathcal{O}_{k,v}).$$

Then $R_0^{\mathsf{T}} R_0 = \alpha I_4$. If $4 \mid (m_1 - m_2)$, then for $R = \mathrm{diag}(I_{m_1}, R_0^{-1^{[(m_1-m_2)/4]}}, I_{n_2})$, $R^{\mathsf{T}} Q' R = Q$, where Q, Q' are the matrices define the quadratic forms q, q'.

If instead $2 \mid r_2$ and $2 \mid (m_1 - m_2)$, we claim the existence of $x_v, y_v \in \mathcal{O}_{k,v}$ such that $x_v^2 + y_v^2 = \alpha$, particularly for $v \mid 2$. In fact, the Hilbert reciprocity argument in the proof of

Lemma 4.3 implies $x_v^2+y_v^2=\alpha$ for certain $x_v, y_v \in k_v$. Since $k_v/\mathbb{Q}_2$ is unramified, $v_2(x_v^2+y_v^2) \leq \min\{2v_2(x_v), 2v_2(y_v)\}+1$, which forces x_v, y_v to be integers. Thus, we can take

$$R_0 = \begin{pmatrix} x_v & -y_v \\ y_v & x_v \end{pmatrix},$$

and $R = \mathrm{diag}(I_{m_1}, R_0^{-1^{[(m_1-m_2)/2]}}, I_{n_2})$.

In both cases, $x \mapsto Rx$ gives a isomorphism $f_v : (k_v^d, q) \xrightarrow{\sim} (k_v^d, q')$ between quadratic space. Since $x_v, \ldots, w_v \in \mathcal{O}_v$, and α is a unit, we have $R \in \mathrm{GL}(d, \mathcal{O}_{k,v})$.

The matrix R further induces the isomorphism between Clifford algebras, $\varphi_v : \mathrm{Cl}^+(k_v^d, q) \xrightarrow{\sim} \mathrm{Cl}^+(k_v^d, q')$, restricted to the algebraic group isomorphism $\mathbf{G}(k_v) \xrightarrow{\sim} \mathbf{G}'(k_v)$. All isomorphisms at finite places together give $\Phi : \mathbf{G}(\mathbb{A}_k^f) \xrightarrow{\sim} \mathbf{G}'(\mathbb{A}_k^f)$. Since $R \in \mathrm{GL}(d, \mathcal{O}_{k,v})$, the induced Clifford algebra isomorphism φ_v sends $\mathscr{C}^+(k_v^d, q)$ to $\mathscr{C}^+(k_v^d, q')$. Consequently, φ_v induces an isomorphism between the lattices Γ_v and Γ_v'. Therefore, Φ gives an isomorphism between the compact open subgroups,

$$\prod_v \Gamma_v \cong \prod_v \Gamma_v'.$$

In the unitary case, we claim there exists $\xi_v, \zeta_v \in \mathcal{O}_{E,v} = \mathcal{O}_E \otimes_{\mathcal{O}_k} \mathcal{O}_{k,v}$, such that

$$N_{E_v/k_v}(\xi_v) + N_{E_v/k_v}(\zeta_v) = \xi_v\overline{\xi}_v + \zeta_v\overline{\zeta}_v = \alpha.$$

This follows our two-square expression in (1) for each $v \nmid 2$; for $v \mid 2$, since $a \in k$ is assumed to be a unit, the existence of x_v, y_v, z_v and $w_v \in \mathcal{O}_{k,v}$ with $x_v^2 - ay_v^2 + z_v^2 - aw_v^2 = \alpha$ is again guaranteed by [HH23, Proposition 4.3]. This implies that $\xi_v = x_v + y_v\sqrt{a}$ and $\zeta_v = z_v + w_v\sqrt{a} \in \mathcal{O}_{E,v}$ satisfy our claim.

Now, we take the matrix

$$R_0 = \begin{pmatrix} \xi_v & -\overline{\zeta}_v \\ \zeta_v & \overline{\xi}_v \end{pmatrix} \in \mathrm{Mat}_2(\mathcal{O}_{E,v}),$$

then $R_0^* R_0 = \alpha I_2$ and $\det(R_0) = \alpha$. Define $R = \mathrm{diag}(I_{m_1}, R_0^{-1^{[(m_1-m_2)/2]}}, I_{n_2})$, then $R \in \mathrm{GL}(d, \mathcal{O}_{E,v})$, and $R^* H' R = H$, where H, H' are the matrices defining the Hermitian forms h, h'. This gives the algebraic group isomorphism,

$$\varphi_v : \mathbf{G}(k_v) \xrightarrow{\sim} \mathbf{G}'(k_v), \quad X \mapsto RXR^{-1}.$$

The remainer of the proof is similar to the spin case.

In the symplectic case, since the reduced norm map $\mathrm{Nrd} : \mathcal{O}_{B,v}^\times \to \mathcal{O}_{k,v}^\times$ is surjective, there exists $\xi_v \in \mathcal{O}_{B,v}$ such that $\mathrm{Nrd}(\xi_v) = \alpha$. Define $R = \mathrm{diag}(I_{m_1}, \xi_v^{-1} I_{m_1-m_2}, I_{n_2})$, then $R \in \mathrm{GL}(d, \mathcal{O}_{B,v})$ and $R^* H' R = H$. The remainder of the proof is again similar to the spin and unitary cases.

Case (2): Exceptional groups. Recall that we defined in the proof of Proposition 3.10 that

$$C = (a, b, c)_k,\ C' = (a, b, c')_k,\ A = H_3(C),\ A' = H_3(C'),$$

where $a, b, c, c' \in \mathcal{O}_k^\times$. Let $Q = (a, b)_k$, and we fix a maximal $\mathcal{O}_k$-order $\mathcal{O}_Q$. For each finite place v, there exists $\xi_v \in \mathcal{O}_{Q,v}$ such that $\mathrm{Nrd}(\xi_v) = c/c'$; since c, c' are units, ξ_v is a unit in $\mathcal{O}_{Q,v}$. Denote $C = Q \oplus Q\ell$ and $C' = Q \oplus Q\ell'$, where $\ell^2 = c$ and $\ell'^2 = c'$; at each finite place v, define an isomorphism,

$$\psi_v : C_v \to C_v', \quad \psi_v(x + y\ell) = x + \xi_v y\ell.$$

It is clear that ψ_v identifies $\mathcal{O}_{C,v}$ with $\mathcal{O}_{C',v}$, where $\mathcal{O}_C \coloneqq \mathcal{O}_Q \oplus \mathcal{O}_Q\ell$ and $\mathcal{O}_C' \coloneqq \mathcal{O}_Q \oplus \mathcal{O}_Q\ell'$. This induces an isomorphism between the corresponding Albert algebras:

$$\psi_v(A) = A', \quad \psi_v(\mathcal{O}_A) = \mathcal{O}_{A'},$$

where $\mathcal{O}_A = H_3(\mathcal{O}_C)$, and $\mathcal{O}_{A'} = H_3(\mathcal{O}_{C'})$. Thus isomorphism between the algebraic groups,

$$\varphi_v :\ G(A_v) = \mathbf{G}(k_v) \xrightarrow{\sim} \mathbf{G}'(k_v) = G(A_v').$$

For the lattices $\Lambda \subset W$, $\Lambda' \subset W'$, and the isomorphism f_v satisfying (1), we discuss the E_7 and E_8 cases separately. In the E_7 case, realizing

$$W = (k \oplus A) \otimes_k B, \quad W' = (k \oplus A') \otimes_k B,$$

then the linear isomorphism $f_v : W_v \to W_v'$ can be taken as

$$f_v = (\mathrm{id}_{k_v} \oplus \psi_v) \otimes \mathrm{id}_{B_v}.$$

The isomorphism f_v is compatible with φ_v, giving the first claim in (1). Define

$$\Lambda = (\mathcal{O}_k \oplus \mathcal{O}_A) \otimes_{\mathcal{O}_k} \mathcal{O}_B, \quad \Lambda' = (\mathcal{O}_k \oplus \mathcal{O}_{A'}) \otimes_{\mathcal{O}_k} \mathcal{O}_B,$$

then the second claim in (1) is guaranteed.

In the E_8 case, realizing the Lie algebras

$$W = \mathrm{Lie}(G(A, C)) = \mathrm{Der}(A) \oplus \mathrm{Der}(C) \oplus (A_0 \otimes_k C_0),$$

and

$$W' = \mathrm{Lie}(G(A', C)) = \mathrm{Der}(A') \oplus \mathrm{Der}(C) \oplus (A_0' \otimes_k C_0),$$

where A_0 and C_0 mean the corresponding trace-free subalgebras. The linear isomorphism,

$$f_v = \mathrm{d}\,\varphi_v = (\psi_v)_* \oplus \mathrm{id} \oplus (\psi_v \otimes \mathrm{id}),$$

naturally satisfies the first claim in (1). Take

$$\Lambda = \mathrm{Der}(\mathcal{O}_A) \oplus \mathrm{Der}(\mathcal{O}_C) \oplus (\mathcal{O}_{A_0} \otimes_{\mathcal{O}_k} \mathcal{O}_{C_0}),\ \Lambda' = \mathrm{Der}(\mathcal{O}_{A'}) \oplus \mathrm{Der}(\mathcal{O}_C) \oplus (\mathcal{O}_{A_0'} \otimes_{\mathcal{O}_k} \mathcal{O}_{C_0}),$$

then the second claim in (1) also holds.

Now, for $g \in \Gamma$, (1) gives

$$\rho'(\varphi_v(g))\Lambda' = \rho'(\varphi_v(g)) f_v \Lambda = f_v \rho(g) \Lambda = f_v \Lambda = \Lambda',$$

and similarly $(\rho'(\varphi_v(g)) - \mathrm{id}_{W'})\Lambda' \subseteq 3\Lambda'$. Converse relations also hold, thus φ_v restricts to an isomorphism between Γ_v and Γ_v'. The product isomorphism Φ thus restricts to an isomorphism between $\prod_v \Gamma_v$ and $\prod_v \Gamma_v'$, completes our proof. □

5. Kählerness and Profinite Rigidity of Arithmetic Lattices

Based on the results developed in the previous sections, we construct two families of finitely presented and residually finite groups G_1 and G_2 so that they have the same profinite completions $\widehat{G}_1 \cong \widehat{G}_2$, moreover G_1 is a Kähler group and G_2 is non-Kähler.

5.1. **The Kähler lattice.** First of all, we recall the following proposition originally due to Guichardet and Wigner but reformulated by Wienhard:

Proposition 5.1 ([Wie05, Chapter 3, Proposition 1.2],[GW78])**.** *Let $X = G/K$ be an irreducible Riemannian symmetric space, where K is a maximal compact subgroup of a connected semisimple, with finite center and no compact factors, real Lie group G. Then X is Hermitian if it satisfies one of the following equivalent properties:*

1. *X admits an $\mathrm{Isom}(X)^\circ$-invariant Kähler form;*
2. *X carries an $\mathrm{Isom}(X)^\circ$-invariant complex structure;*
3. *The center $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k}$ is nontrivial, where $\mathfrak{k}$ is the Lie algebra of K.*

Here $\mathrm{Isom}(X)^\circ$ is the identity component of the isometry group of X.

Proposition 5.2. *Let Γ_1 be a torsion-free cocompact lattice in $\prod_{i=1}^k \mathrm{Spin}(n_i, 2)^{r_i}$ for integers $n_i \geq 3, r_i \geq 1$. Then Γ_1 is a Kähler group.*

Proof. For any integer $n_i \geq 3$, denote by K_i a maximal compact subgroup of $\mathrm{Spin}(n_i, 2)$ and $X_i \coloneqq \mathrm{Spin}(n_i, 2)/K_i$ the irreducible symmetric space of $\mathrm{Spin}(n_i, 2)$. Identifying

$$\prod_{i=1}^k X_i^{r_i} \cong (\prod_{i=1}^k \mathrm{Spin}(n_i, 2)^{r_i}) / (\prod_{i=1}^k K_i^{r_i}),$$

the product $\prod_{i=1}^k \mathrm{Spin}(n_i, 2)^{r_i}$ acts on $\prod_{i=1}^k X_i^{r_i}$ properly. Thus, the action of the lattice $\Gamma_1 < \prod_{i=1}^k \mathrm{Spin}(n_i, 2)^{r_i}$ on $\prod_{i=1}^k X_i^{r_i}$ is properly discontinuous. Moreover, for any $x \in \prod_{i=1}^k X_i^{r_i}$, the stabilizer $\mathrm{Stab}(x) < \prod_{i=1}^k \mathrm{Spin}(n_i, 2)^{r_i}$ is compact. The discreteness of Γ_1 then implies that $\Gamma_1 \cap \mathrm{Stab}(x)$ is finite; as Γ_1 is torsion-free, this intersection is indeed trivial. Thus, the action of Γ_1 on $\prod_{i=1}^k X_i^{r_i}$ is free.

Next, consider the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $\mathrm{Spin}(n_i,2)$ or $\mathrm{SO}_0(n_i,2)$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k} = \mathfrak{so}(n_i) \oplus \mathfrak{so}(2)$ is the Lie algebra of K_i. Note that the center of $\mathfrak{k}$ is

$$\mathfrak{c}(\mathfrak{k}) = \mathfrak{c}(\mathfrak{so}(n_i)) \oplus \mathfrak{c}(\mathfrak{so}(2)) = \mathfrak{so}(2) \neq 0.$$

since $\mathfrak{so}(n_i)$ has trivial center for $n_i \geq 3$ and $\mathfrak{so}(2)$ is $\mathbb{R}$. Applying Proposition 5.1, we then have $\mathrm{Spin}(n_i,2)$ is of Hermitian type, and X_i admits an $\mathrm{Isom}(X_i)^\circ$-invariant complex structure J_i and Kähler form ω_i. The natural action of $\mathrm{Spin}(n_i,2)$ on X_i factors through the adjoint quotient

$$\rho_i : \mathrm{Spin}(n_i,2) \longrightarrow \mathrm{PSO}^+(n_i,2) \cong \mathrm{Isom}(X_i)^\circ.$$

Hence the complex structure J_i and the Kähler form ω_i are $\mathrm{Spin}(n_i,2)$-invariant. The product of these Kähler structures gives the Kähler structure on $\prod_{i=1}^k X_i^{r_i}$, which is $\prod_{i=1}^k \mathrm{Spin}(n_i,2)^{r_i}$-invariant. As a lattice, Γ_1 thus also preserves the complex structure and Kähler form on the product space.

Since Γ_1 is a cocompact lattice of $\prod_{i=1}^k \mathrm{Spin}(n_i,2)^{r_i}$, $\Gamma_1 \backslash \prod_{i=1}^k X_i^{r_i}$ is a compact Kähler manifold. The free and properly discontinuous action of Γ_1 on the simply connected space $\prod_{i=1}^k X_i^{r_i}$ implies

$$\pi_1(\Gamma_1 \backslash \prod_{i=1}^k X_i^{r_i}) \cong \Gamma_1.$$

Therefore, Γ_1 is a Kähler group. □

5.2. **The non-Kähler lattices.** In this subsection, we show that any torsion-free cocompact arithmetic lattice Γ_2 of products of $\mathrm{Spin}(n,1)$ ($n \geq 3$) is non-Kähler. We first give some necessary concepts and tools used in the discussion on the non-Kählerness of Γ_2. Recall that a *hyperbolic 2-orbifold* is a quotient $\Sigma = \Lambda \backslash \mathbb{H}^2$. Here $\Lambda < \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}(\mathbb{H}^2)^\circ$ is discrete and cocompact; in this case, $\Lambda \cong \pi_1^{\mathrm{orb}}(\Sigma)$. We have the following important theorem due to Delzant-Py.

Theorem 5.3 ([DP12, Theorem 1, Corollary 1])**.** *Let $\Gamma = \pi_1(M)$ be a Kähler group corresponding to a compact Kähler manifold M. If for some $n \geq 3$, a homomorphism $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ has Zariski-dense image, then there exist a hyperbolic 2-orbifold Σ, such that ρ factors through the map $f_* : \pi_1(M) \longrightarrow \pi_1^{\mathrm{orb}}(\Sigma)$ induced by a holomorphic fibration $f : M \to \Sigma$.*

We also need the following simple lemma on the cohomological dimension for discrete subgroups of Lie groups (See e.g., [Bro82, Chapter VIII, Section 9.4]).

Lemma 5.4. *Let Γ be a torsion-free discrete subgroup of a real Lie group G of non-compact type. Suppose G has finitely many connected components. We have*

$$\mathrm{cd}(\Gamma) \leq \dim_{\mathbb{R}} X,$$

where X is the symmetric space corresponding to G. Moreover, the equality holds if and only if Γ is cocompact in G.

Proof. For the maximal compact subgroup K of G, by the Second Manifold Splitting Theorem [HN12, Theorem 14.3.11], we have $X \coloneqq G/K$ is diffeomorphic to some $\mathbb{R}^m$, where $m = \dim_{\mathbb{R}} G - \dim_{\mathbb{R}} K$. Therefore, X/Γ is a Eilenberg–MacLane space $K(\Gamma,1)$. Then the lemma follows from [Bro82, Chapter VIII, Proposition 8.1-(a)], which says that for any d-dimentional aspherical manifold Y, $\mathrm{cd}(\pi_1(Y)) \leq d$, with equality if and only if Y is closed. □

Theorem 5.5. *Let G be a connected semisimple real Lie group with finite center and no compact factors, with a factor isomorphic to $\mathrm{Spin}(n,1)$, $n \geq 4$ and is even. Then any torsion-free, irreducible and cocompact lattice $\Gamma_2 < G$ is not a Kähler group.*

We make the following observation before the proof of Proposition 5.5. Suppose $G = \mathrm{Spin}(n,1) \times H$, where the product H of the other factor groups is possibly trivial. For even n, the natural projection map

$$\mathrm{SO}(n,1) \longrightarrow \mathrm{PO}(n,1) \cong \mathrm{Isom}(\mathbb{H}^n)$$

is an isomorphism, since $-I_{n+1} \notin \mathrm{SO}(n,1)$. Hence the standard double covering induces

$$\theta_0 : \mathrm{Spin}(n,1) \longrightarrow \mathrm{SO}_0(n,1) < \mathrm{Isom}(\mathbb{H}^n),$$

with kernel $\{\pm 1\}$. Let θ_H be the natural quotient map $H \to H/Z(H) \coloneqq H_{\mathrm{ad}}$, and consider the product homomorphism $\theta \coloneqq \theta_0 \times \theta_H$. Since Γ_2 is a torsion-free subgroup of $\mathrm{Spin}(n,1) \times H$, we have $\Gamma_2 \cap \ker\theta = \{1\}$, i.e.,

$$\theta|_{\Gamma_2} \colon \Gamma_2 \to \mathrm{SO}_0(n,1) \times H_{\mathrm{ad}}$$

is injective. Since Γ_2 is a cocompact subgroup of $\mathrm{Spin}(n,1)\times H$ and $\theta_0 : \mathrm{Spin}(n,1) \to \mathrm{SO}_0(n,1)$ is a double cover, one can view $\Gamma_2 \cong \theta(\Gamma_2)$ as a torsion-free cocompact subgroup of $\mathrm{SO}_0(n,1)\times H_{\mathrm{ad}}$. Since θ is a quotient by a finite normal subgroup, Γ_2 is also irreducible in $\mathrm{SO}_0(n,1) \times H_{\mathrm{ad}}$. We require the following lemma.

Lemma 5.6. *For the factor map $\vartheta : \mathrm{SO}_0(n,1) \times H_{\mathrm{ad}} \to \mathrm{SO}_0(n,1)$, $\vartheta(\Gamma_2) < \mathrm{SO}_0(n,1)$ is Zariski dense, and the restriction map $\widetilde{\vartheta} : \Gamma_2 \to \mathrm{SO}_0(n,1)$ is injective.*

Proof. For our first claim, if H is trivial, then ϑ is the identity map, and the conclusion comes from the Borel density theorem [Bor60]. In other cases, the irreducibility of Γ_2 implies the density of every $\vartheta(\Gamma_2)$ under the usual topology, and the Zariski density follows.

For our second claim, the conclusion is instant when H is trivial. In other cases, it suffices to show that $T \coloneqq \ker(\widetilde{\vartheta}) = \Gamma_2 \cap (\{1\} \times H_{\mathrm{ad}})$ is trivial. In fact, consider the map

$$\widetilde{\vartheta}' \colon \Gamma_2 \hookrightarrow \mathrm{SO}_0(n,1) \times H_{\mathrm{ad}} \to H_{\mathrm{ad}}$$

where the second map is the natural projection onto the H_{ad} factor. The subgroup $\widetilde{\vartheta}'(T)$ is normal in $\widetilde{\vartheta}'(\Gamma_2)$. Since T is discrete and $\widetilde{\vartheta}'|_T$ is injective, $\widetilde{\vartheta}'(T)$ is discrete, hence closed in H_{ad}. Since Γ_2 is irreducible, $\widetilde{\vartheta}'(\Gamma_2)$ is dense in H_{ad}, it follows that $\widetilde{\vartheta}'(T)$ is normal in H_{ad}. By [Kna96, Proposition 1.93(d)], $\widetilde{\vartheta}'(T)$ is contained in the center of H_{ad}, which is trivial. We deduce that $T = \{1\}$ from these facts. □

Proof of Theorem 5.5. Suppose by contradiction that Γ_2 is a Kähler group. Note n is even, so there is a homomorphism induced from Lemma 5.6:

$$\rho \colon \Gamma_2 \xrightarrow{\widetilde{\vartheta}} \mathrm{SO}_0(n,1) \hookrightarrow \mathrm{SO}(n,1) \cong \mathrm{Isom}(\mathbb{H}^n),$$

so that the image of ρ is Zariski dense since the Zariski closure $\overline{\mathrm{SO}_0(n,1)} = \mathrm{SO}(n,1)$. Implied by Theorem 5.3, ρ factors through a map $\Gamma_2 \to \pi_1^{\mathrm{orb}}(\Sigma)$. The injectivity of ρ implies that $\Gamma_2 < \pi_1^{\mathrm{orb}}(\Sigma)$, hence $\mathrm{cd}(\Gamma_2) \le 2$.

On the other hand, as a cocompact torsion-free discrete lattice in $\mathrm{SO}_0(n,1) \times H_{\mathrm{ad}}$, Lemma 5.4 implies $\mathrm{cd}(\Gamma_2) = \dim_{\mathbb{R}}\big(\mathbb{H}^n \times (X/K_H)\big) \ge n \ge 4$, a contradiction. Hence Γ_2 is not a Kähler group. □

Proposition 5.7. *Γ_1 and Γ_2 are finitely presented and residually finite.*

Proof. Since Γ_1 and Γ_2 are torsion-free cocompact lattices, they are the fundamental groups of the compact manifolds. Hence, both groups are finitely presented. Being finitely generated linear groups over a field of characteristic zero, they are residually finite by Mal'cev's theorem. □

Theorem 5.8. *Let $n \ge 2$, $r_1 \ge 0$, and $r_2 \ge 1$ be integers such that $2r_1 + r_2 \ge 2$ and either n is odd or r_2 is even. There exist finitely presented, residually finite, torsion-free cocompact arithmetic lattices*

$$\Gamma_1 < \mathrm{Spin}(2n-1,2)^{r_1+r_2}, \Gamma_2 < \mathrm{Spin}(2n-1,2)^{r_1} \times \mathrm{Spin}(2n,1)^{r_2},$$

with isomorphic profinite completions, $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$, such that Γ_1 is Kähler and Γ_2 is non-Kähler. In addition, if $r_1 \ge 1$, Γ_1 and Γ_2 can be explicitly taken as certain level-3 congruence subgroups of the corresponding integral spin groups.

Proof. We first show $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ by applying Theorem 4.1-(1). Here we have the signature pairs $(m_1, n_1) = (2n-1, 2)$ and $(m_2, n_2) = (1, 2n)$. Note $d = 2n+1 \geq 5$, and $2r_1 + r_2 \geq 2$ implies the required rank condition. Now if n is odd, then $4 \mid (2n-1)-1$; if r_2 is even, then $2 \mid (2n-1)-1$, hence the congruence conditions hold. So it follows from Theorem 4.1-(1), there exist torsion-free cocompact arithmetic lattices $\Gamma_1 < \mathrm{Spin}(2n-1,2)^{r_1+r_2}$, $\Gamma_2 < \mathrm{Spin}(2n-1,2)^{r_1} \times \mathrm{Spin}(2n,1)^{r_2}$ with $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$. If $r_1 \geq 1$, Theorem 4.8 ensure that Γ_1, Γ_2 can be taken as certain level-3 congruence subgroups.

Now, directly from Proposition 5.2 and Theorem 5.5, we can see that Γ_1 is Kähler and Γ_2 is non-Kähler. Moreover, by Proposition 5.7, Γ_1, Γ_2 are finitely presented and residually finite. □

We conclude the paper with the following concrete example of lattices,

$$\Gamma_1 < \mathrm{Spin}(3,2)^3 \cong \mathrm{Sp}(4,\mathbb{R})^3, \quad \Gamma_2 < \mathrm{Spin}(3,2) \times \mathrm{Spin}(4,1)^2 \cong \mathrm{Sp}(4,\mathbb{R}) \times \mathrm{Sp}(1,1)^2.$$

Example 5.9. *Let α be a root of $f(x) = x^4 - 4x^2 + x + 1$, and put $k = \mathbb{Q}(\alpha)$. The properties of f imply that both α and $\alpha - 1$ are algebraic units and that $k_v/\mathbb{Q}_2$ is unramified at the unique place $v \mid 2$. Moreover, α lies in $(1, \infty)$ at one real place, in $(0, 1)$ at one real place, and is negative at the other two. Define*

$$q_1 = \langle 1^{[3]}, (\alpha-1)^{[2]} \rangle, \qquad q_2 = \langle 1, \alpha^{[2]}, (\alpha-1)^{[2]} \rangle,$$

and let $\mathbf{G}_i = \mathrm{Spin}(k, q_i)$ for $i = 1, 2$. The products of their noncompact archimedean factors are

$$G_1 \cong \mathrm{Spin}(3,2)^3, \qquad G_2 \cong \mathrm{Spin}(3,2) \times \mathrm{Spin}(4,1)^2.$$

These forms satisfy the hypotheses of Lemma 4.3 and Proposition 4.4 for the second alternative of Theorem 4.1(1).

Let $\Gamma_i \leq \mathbf{G}_i(k)$ be the level-3 congruence subgroups defined in Subsection 4.2. Moreover, $\mathrm{disc}(f) = 1957$ *is square-free, so $\mathcal{O}_k = \mathbb{Z}[\alpha]$ in this definition. Identifying these congruence subgroups with their images under the archimedean projections, we obtain arithmetic lattices*

$$\Gamma_1 < \mathrm{Spin}(3,2)^3, \qquad \Gamma_2 < \mathrm{Spin}(3,2) \times \mathrm{Spin}(4,1)^2.$$

Since both G_1 and G_2 have a factor $\mathrm{Spin}(3,2)$*, Proposition 4.8 implies that Γ_1 and Γ_2 have isomorphic profinite completions. All factors of G_1 are* $\mathrm{Spin}(3,2)$*, whereas G_2 has factors* $\mathrm{Spin}(4,1)$*. Thus Proposition 5.2 and Theorem 5.5 imply that Γ_1 is Kähler, whereas Γ_2 is not.*

AI Declaration The mathematical ideas and proofs of this paper are from the authors. AI was only used for typesetting and language clean-up. We take the responsibility for the content.

Appendix A. Overview of the Noncompact Real Forms of Lie Groups of Dynkin types A_l, B_l, C_l, and D_l

To demonstrate that Proposition 3.6, Proposition 3.13 and Theorem 3.15 fully characterizes the profinite rigidity for lattices in real forms of Lie groups of types A_l, B_l, C_l, and D_l of rank ≥ 2, we summarize the groups in the following table, and provide the relevant information of them. The classification and the symmetric space dimensions are standard; for the inner and outer forms of each Lie group, see the description of the classical absolutely simple groups in [Tit66b]. Note in the Table below, ℓ denotes the absolute rank, that is, the index in the absolute Dynkin type; it should not be confused with the real rank. For a family parametrized by a signature (m, n), we write $d = m + n$.

Type	Cartan type	Noncompact real form	$\dim(G/K)$	Inner to compact form?
A_l	AI	$\mathrm{SL}(l+1,\mathbb{R})$	$\frac{l(l+3)}{2}$	No
	AII	$\mathrm{SL}\left(\frac{l+1}{2},\mathbb{H}\right)$	$\frac{(l-1)(l+2)}{2}$	No
	AIII	$\mathrm{SU}(m,n), \quad m+n=l+1$	$2mn$	Yes
B_l	BI	$\mathrm{Spin}(m,n), \quad m+n=2l+1$	mn	Yes
C_l	CI	$\mathrm{Sp}(2l,\mathbb{R})$	$l(l+1)$	Yes
	CII	$\mathrm{Sp}(m,n), \quad m+n=l$	$4mn$	Yes
D_l	DI	$\mathrm{Spin}(m,n), \quad m+n=2l$	mn	Yes if m,n are even, No if m,n are odd
	DIII	$\mathrm{Spin}^*(2l)$	$l(l-1)$	Yes

TABLE 1. Noncompact real forms of the classical simple Lie groups and the dimensions of their associated symmetric spaces.

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Department of Mathematics, University of Georgia, Athens, GA 30603
Email address: yukun.du@uga.edu
URL: https://benniedu.github.io/

School of Mathematics, Shandong University, Jinan, 250100, P. R. China.
Email address: feng.hao@sdu.edu.cn

School of Mathematics, Hunan University, Changsha, Hunan, 410082, P.R.China
Email address: kzhumath@gmail.com
URL: https://sites.google.com/view/kejiazhu