LATTICES OF MINIMAL COVOLUME IN SL_2 OVER DYADIC LOCAL FIELDS

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Abstract. In this paper, we classify the lattices of minimal covolume in $\mathrm{SL}(2,K)$ for each finite extension $K/\mathbb{Q}_2$. This extends a result of Lubotzky and Weigel, which treats finite extensions of $\mathbb{Q}_p$ for odd primes p.

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1. Introduction

1.1. **Background.** It is a classical problem to determine how *dense* a lattice Γ in a Lie group G can be. More precisely, one seeks the minimum of the normalized Haar measure $\mu(\Gamma\backslash G)$. When $G=\mathrm{SL}(2,\mathbb{R})$, a result of Siegel [11] implies that the minimum is attained when Γ is the $(2,3,7)$-triangle group, where $\min_\Gamma \mu(\Gamma\backslash G)=\pi/21$. The complex case $G=\mathrm{SL}(2,\mathbb{C})$ was addressed more recently by Gehring-Martin [5] and Marshall-Martin [8]. Namely, they determined that

$$\min_\Gamma \mu(\Gamma\backslash \mathrm{SL}(2,\mathbb{C})) = \frac{275^{3/2}}{128\pi^6}\zeta_k(2) \approx 0.03905\ldots,$$

where $k=\mathbb{Q}(\alpha)$ and α is a nonreal root of $\alpha^4+6\alpha^3+12\alpha^2+9\alpha+1=0$. This minimal volume is attained by two $\mathbf{H}^3$-orbifolds; both admit 60-sheeted manifold covers, given by the $-4/3$-Dehn surgery along the $(-2,3,7)$ pretzel knot and the $(-7/2,-10/3)$-Dehn surgery along the Whitehead link, respectively [3]. By contrast, Margulis' arithmeticity theorem implies that lattices in higher-rank Lie groups are arithmetic, and lattices of minimal covolume among them admit more succinct descriptions. In particular, Thilmany [12] proved that the minimal covolume lattice in $\mathrm{SL}(n,\mathbb{R})$ is $\mathrm{SL}(n,\mathbb{Z})$ for $n\geq 3$; for $\mathrm{Sp}(2n,\mathbb{R})$ with $n\geq 2$, Džambić, Holm, and Köhl [4] showed that the lattice of minimal covolume is $\mathrm{Sp}(2n,\mathbb{Z})$.

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An analogous question arises for Lie groups over non-archimedean local fields. Lubotzky [6] studied the minimal covolume lattices in $\mathrm{SL}(2,K)$, where K is either a field of positive characteristic $\mathbf{F}_q((t))$ or the p-adic field $\mathbb{Q}_p$. Notably, the minimal covolume lattices in the positive-characteristic cases are non-uniform lattices $\mathrm{SL}_2(\mathbb{F}_q[t])$. In the higher-rank case, Allcock and Kato [1] determined the lattices of minimal covolume in $\mathrm{PGL}(3,\mathbb{Q}_2)$, with covolume $1/21$. The two minimizing lattices are commensurable, arising from a Hermitian form over $\mathbb{Q}(\sqrt{-7})$.

The case where K is a finite extension of $\mathbb{Q}_p$ is more interesting. When p is an odd prime, Lubotzky and Weigel [7] determined the lattices of minimal covolume in $\mathrm{SL}(2,K)$. The lattices they constructed are described by graphs of finite groups with up to six vertices, whereas paths of length 1 and loops of length 2 are enough when $K=\mathbb{Q}_p$.

A few questions arise when generalizing Lubotzky and Weigel's method to the $p=2$ case. For one thing, when $p=2$, elliptic elements in $\mathrm{SL}(2,K)$ of order p^k have more intricate dynamics, in particular, a more complicated description of the diameter of the fixed subtree. Furthermore, many vertex groups of lattices in $\mathrm{SL}(2,K)$ are dicyclic, and the elements of order 4 in these groups result in more complicated behavior when $p=2$.

We aim to resolve these issues and settle the problem for SL_2 over finite extensions of $\mathbb{Q}_2$. In Section 2, we generalize [7, Theorem 4.2] to give a complete description of the fixed subtrees of elements in $\mathrm{SL}(2,K)$ of order 2^k. In Section 2, we discuss the possible vertex groups of $\mathrm{SL}(2,K)$ lattices, including the dicyclic and binary polyhedral ones. We construct the candidate lattices of minimal covolume in Section 4, and complete the proof of our main theorem in Section 5.

1.2. **Notation and Previous Results.** In this paper, unless otherwise stated, we let K be a finite extension of $\mathbb{Q}_2$, with uniformizer $\pi=\pi_K$ and ring of integers $\mathcal{O}=\mathcal{O}_K$. The residue field $\mathcal{O}_K/\pi_K\mathcal{O}_K$ is isomorphic to a finite field $\mathbb{F}_q$, $q=2^f$, where $f=f(K/\mathbb{Q}_2)$ is called the *extension degree* of $K/\mathbb{Q}_2$. With the *valuation* normalized by $v_K(\pi_K)=1$, $e=e(K/\mathbb{Q}_2)=v_K(2)$ is called the *ramification degree*. These quantities satisfy

$$[K:\mathbb{Q}_2]=e(K/\mathbb{Q}_2)\cdot f(K/\mathbb{Q}_2).$$

The *Bruhat-Tits tree* of the non-archimedean Lie group $G=\mathrm{SL}(2,K)$ is the quotient $T=\mathrm{SL}(2,K)/\mathrm{SL}(2,\mathcal{O}_K)$, which is realized as a $(q+1)$-regular tree with edge lengths normalized to 1. For any vertex $x\in\mathcal{V}(T)$, the *vertex stabilizer* G_x is isomorphic to $\mathrm{SL}(2,\mathcal{O}_K)$. For any element $g\in\mathrm{SL}(2,K)$ or subgroup $A<\mathrm{SL}(2,K)$, we denote its *fixed subtree* by T_g, or T_A, respectively; we note that $T_g=T_{\langle g\rangle}$.

For the vertex groups appearing in this paper, we denote the dicyclic group of order $4m$ by $C_{2m}.2$, which is a double cover of the dihedral group D_{2m}. We denote the three binary polyhedral groups by $2.A_4$, $2.S_4$ and $2.A_5$, which are double covers of the polyhedral groups A_4, S_4, and A_5, respectively.

In the algebraic closure $\overline{\mathbb{Q}}_2$, we let ξ_m be the m-th root of unity, and $z_m=\xi_m+\xi_m^{-1}$. In particular, $\xi_3=\frac{-1+\sqrt{-3}}{2}$, $\xi_4=\sqrt{-1}$, $z_5=\frac{-1+\sqrt{5}}{2}$, $z_8=\sqrt{2}$, and $z_{12}=\sqrt{3}$.

For an element g of order 2^k, its fixed subtree T_g is related to the ramification number of K. In particular, [7, Theorem 4.2] provides the following:

Theorem 1.1 ([7, Theorem 4.2], the case $p = 2$)**.** *Let K be a finite extension of $\mathbb{Q}_2$, and let $g \in \mathrm{SL}(2, K)$ have order 2^k, with $k \geq 2$. It follows that g has the roots of unity ξ_{2^k} and $\xi_{2^k}^{-1}$ as eigenvalues, so $z_{2^k} \coloneqq \xi_{2^k} + \xi_{2^k}^{-1} \in K$. Define the ramification number $r_k = e(K/\mathbb{Q}_2(z_{2^k}))$ and put $C = C_{\mathrm{GL}(2,K)}(g)$. Then the fixed subtree is $T_g = N(\mathfrak{C}_g, d_k)$, where $\mathfrak{C}_g$ is the characteristic subtree of C, called the* center *of T_g, and $d_k \in \mathbb{N}$ is called the* width *of T_g. Moreover, one of the following holds.*

- *If the root of unity $\xi_{2^k} \in K$, then C is a split torus, $\mathfrak{C}_g$ is a bi-infinite path, and $d_k = r_k$.*
- *If $\xi_{2^k} \notin K$, then C is a non-split torus, and $|K(\xi_{2^k})/K| = 2$:*
  - *If $e(K(\xi_{2^k})/K) = 1$, i.e., the extension is unramified, then $\mathfrak{C}_g$ is a single vertex, and $d_k = r_k$.*
  - *If $e(K(\xi_{2^k})/K) = 2$, i.e., the extension is totally ramified, then $\mathfrak{C}_g$ is a single edge, and $d_k \geq \frac{r_k-1}{2}$.*

2. Fixed Subtrees of 2-Elements

Lubotzky and Weigel did not further analyze the width behavior in case (2b), especially when r_k is even. Nevertheless, we obtain the following result.

Theorem 2.1. *Let K, g, r_k and C be defined as above, and suppose $\xi_{2^k} \notin K$ and $e(K(\xi_{2^k})/K) = 2$. Then the width d_k of the fixed tree $T_g = N(T_C, d_k)$ satisfies the following.*

- *If r_k is odd, then $d_k = \frac{r_k-1}{2}$.*
- *If r_k is even, then $\frac{r_k}{2} \leq d_k \leq r_k - 1$, and each admissible d_k is attained by some extension K.*

In particular, in case (ii), let $\mathbb{F}_q$ be the residue field of K, let $K_0 \coloneqq \mathbb{Q}_p(\xi_{q-1}, z_{2^k})$, and let π be a uniformizer, so $K = K_0(\pi)$, and $e(K/K_0) = r_k$. Put $w = w_{2^k} \coloneqq 2 - z_{2^k}$, so w is a uniformizer of K_0 and $v(w) = r_k v(\pi)$. Suppose

$$f(x) = x^r - w(a_{r-1}x^{r-1} + \cdots + a_1 x + a_0)$$

is an Eisenstein polynomial of π, where $a_{r-1}, \ldots, a_0 \in \mathcal{O}_{K_0}$ and $a_0 \not\equiv 0 \bmod w$. By abuse of notation, we use the same symbols for their residue classes in $\mathcal{O}_{K_0}/(w.\mathcal{O}_{K_0}) = \mathbb{F}_q$. Notice that $\mathbb{F}_q$ has characteristic 2 and every element has a square root. Then the equations

$$a_j = \sum_{i=1}^{\lfloor \log_2(r/(r-j)) \rfloor} (a_{r-2^i(r-j)})^{1/2^i}, \quad j = 1, 3, \ldots, r-1, \ 2 \nmid j,$$

cannot all be satisfied. Let l be the smallest index j such that the equation does not hold. Then the width is $d_k = \frac{r_k+l-1}{2}$.

Lemma 2.2. *Under the assumptions of Theorem 2.1, we have*

$$2d_k + 1 = \max_{\alpha \in \mathcal{O}_K} \frac{v(\alpha^2 - z\alpha + 1)}{v(\pi)} = \max_{\beta \in \mathcal{O}_K} \frac{v(\beta^2 + w\beta + w)}{v(\pi)}.$$

Proof. The group $\mathrm{PSL}(2,K)$ acts on pairs of endpoints (and $\mathrm{PGL}(2,K)$ on triples) in $\partial_\infty T_K = \mathbf{P}^1(K)$. Therefore, up to a suitable $\mathrm{PSL}(2,K)$-conjugation of g, we may assume that a longest path in T_g lies in the geodesic connecting 0 and $\infty \in \mathbf{P}^1(K)$. On the one hand, the length of this path equals $2d_k+1$. On the other hand, since $\mathrm{ord}(g) = 2^k$, it follows that $\mathrm{tr}(g) = z_{2^k}$. Therefore we can assume

$$g = \begin{pmatrix} \alpha & x \\ x^{-1}(\alpha(z-\alpha)-1) & z-\alpha \end{pmatrix},$$

for some α and $x \in K$. Since the stabilizers of the vertices on the path $(0,\infty)$ are

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,K) \,|\, a,d \in \mathcal{O},\ b \in \pi^i\mathcal{O},\ c \in \pi^{-i}\mathcal{O} \right\}, \quad i \in \mathbb{Z},$$

it follows that $\alpha \in \mathcal{O}_K$, and

$$\ell(T_g \cap (0,\infty)) = \frac{v(x) + v(x^{-1}(\alpha(z-\alpha)-1)}{v(\pi)} = \frac{v(\alpha^2 - z\alpha + 1)}{v(\pi)}.$$

The reverse inequality is immediate. For the other equivalent expression, we note

$$\alpha^2 - z\alpha + 1 = \alpha^2 - (2-w)\alpha + 1 = (\alpha-1)^2 + w(\alpha-1) + w,$$

and $\beta = \alpha - 1 \in \mathcal{O}_K$ if and only if $\alpha \in \mathcal{O}_K$.

□

Lemma 2.3 (cf. [7, Proposition 4.4])**.** *We have*

$$r_k v(\pi) \le \max_{\alpha \in \mathcal{O}_K} v(\alpha^2 - z\alpha + 1) \le 2r_k v(\pi).$$

Proof. For the first inequality, we take $\alpha = 1$:

$$\max_{\alpha \in \mathcal{O}_K} v(\alpha^2 - z\alpha + 1) \ge v(2-z) = v(w) = r_k v(\pi).$$

For the second inequality, we note that in the field extension $K(\xi_{2^k})$,

$$(\alpha^2 - z\alpha + 1) = (\alpha - \xi)(\alpha - \xi^{-1}),$$

so

$$v(\alpha^2 - z\alpha + 1) = v(\alpha - \xi) + v(\alpha - \xi^{-1}).$$

Since $\alpha \in K$, the generator $\sigma \in \mathrm{Gal}(K(\xi_{2^k})/K)$ satisfies $\sigma(\alpha - \xi) = \alpha - \xi^{-1}$, so

$$v(\alpha - \xi) + v(\alpha - \xi^{-1}) = 2v(\alpha - \xi).$$

By the non-archimedean triangle inequality:

$$v(\alpha - \xi) = \min(v(\alpha - \xi), v(\alpha - \xi^{-1})) \le v((\alpha - \xi^{-1}) - (\alpha - \xi)) = v(\xi - \xi^{-1}) = r_k v(\pi).$$

Since the inequality holds for every $\alpha \in \mathcal{O}_K \subset K$, we have

$$\max_{\alpha \in \mathcal{O}_K} v(\alpha^2 - z\alpha + 1) \le 2r_k v(\pi).$$

□

Remark 2.4. If K is a finite extension of $\mathbb{Q}_p$ for an odd prime p, the same arguments give

$$\max_{\alpha\in K} v(\alpha^2 - z\alpha + 1) \leq 2v(\xi_{p^k} - \xi_{p^k}^{-1}) = 2v(\xi_{p^k} - 1) = r_k v(\pi),$$

which differs from the case $p = 2$ by a factor of 2 (cf. [7, proof of Theorem 4.2]). Together with the other inequality,

$$\max_{\alpha\in K} v(\alpha^2 - z\alpha + 1) \geq v(2 - z_{p^k}) = v((1 - \xi_{p^k})^2\xi_{p^k}^{-1}) = r_k v(\pi),$$

the equality $\max_{\alpha\in K} v(\alpha^2 - z\alpha + 1) = r_k v(\pi)$ holds, leading to $d_k = \frac{r_k - 1}{2}$. This agrees with the formula in [7].

Lemma 2.5. *Suppose* $\mathrm{char}(\mathbb{F}_q) = 2$*, and let*

$$g(x) = a_{r-1}x^{r-1} + \cdots + a_1 x + a_0 \in \mathbb{F}_q[x],$$

be a fixed polynomial of odd degree $r - 1$ *with* $a_0 \neq 0$*, and let*

$$C(x) = c_{r/2-1}x^{r/2-1} + \cdots + c_1 x + c_0 \in \mathbb{F}_q[x]$$

be any polynomial of degree $r/2 - 1$*. Then the largest* $j \in \mathbb{N}$ *such that*

$$x^j \mid (1 + x^{r/2}C(x) + C^2(x)g(x))$$

for some $C(x) \in \mathbb{F}_q[x]$ *is the smallest odd number such that the equation*

$$a_j = \sum_{i=1}^{\lfloor \log_2(r/(r-j)) \rfloor} (a_{r-2^i(r-j)})^{1/2^i}$$

does not hold.

Proof. Denote the key expression in the lemma by

$$E(x) = 1 + x^{r/2}C(x) + C^2(x)g(x).$$

To simplify the equation, let

$$V(x) = x^{-r/2}g(x)C(x),$$

then

$$V^2(x) + V(x) = x^{-r}g(x)(1 + E(x)).$$

Express $V(x)$ as a Laurent series:

$$V(x) = \sum_{i\leq r/2} v_{-i}x^{-i},\ v_{-i} \in \mathbb{F}_q.$$

Since the terms of degree less than j in $E(x)$ vanish, the x^{-2l}-term of the equation above is

$$v_{-l}^2 + v_{-2l} = a_{r-2l} \implies v_{-l} = v_{-2l}^{1/2} + a_{r-2l}^{1/2},$$

for each $l > (r - j)/2$. Since $2l > l$, iteration gives

$$v_{-l} = \sum_{i=1}^{\lfloor \log_2(r/l) \rfloor} a_{r-2^i l}^{1/2^i}.$$

Now, for each odd number $l \leq j$, whether the x^l-term vanishes in $E(x)$ is equivalent to whether the x^{l-r}-coefficient equation above holds with the $E(x)$ component omitted. Since

the x^{l-r}-term does not appear in $V^2(x)$, this is equivalent to whether $v_{l-r} = a_l$. Substituting the expression of v_{-l} in terms of coefficients a_i yields the desired conclusion.

□

Proof of Theorem 2.1. By Lemma 2.2, it suffices to compute $\max_{\beta\in\mathcal{O}_K} v(\beta^2 + w\beta + w)$.

If r_k is odd, then

$$v(\beta^2) = 2v(\beta) \neq r_k v(\pi) = v(w).$$

Therefore,

$$v(\beta^2 + w\beta + w) = \min(v(\beta^2), v(w)) \leq r_k v(\pi),$$

and equality holds when $v(\beta) > v(w)/2$, e.g., when $\beta = 0$. Therefore

$$2d_k + 1 = \max_{\beta\in\mathcal{O}_K} \frac{v(\beta^2 + w\beta + w)}{v(\pi)} = r_k,$$

and $d_k = \frac{r_k-1}{2}$.

If r_k is even, by Lemma 2.4,

$$r_k \leq 2d_k + 1 \leq 2r_k,$$

so

$$\frac{r_k - 1}{2} \leq d_k \leq \frac{2r_k - 1}{2}.$$

Since d_k is an integer, we further have

$$\frac{r_k}{2} \leq d_k \leq r_k - 1.$$

To show that each integer in this range occurs, we consider the Eisenstein polynomial

$$x^r + wx^{r/2} + wx^j + w = 0, \quad j = 1, 3, \ldots, r-1,\ 2 \nmid j,$$

and let π be a root of this polynomial. Set $K_0 = \mathbb{Q}_p(\xi_{q-1}, z_{2^k})$, and $K = K_0(\pi)$. Let $\beta = \pi^{r/2}$. Then

$$v(\beta^2 + w\beta + w) = v(\pi^r + w\pi^{r/2} + w) = v(w\pi^j) = (r_k + j)v(\pi),$$

so $d_k \geq \frac{r_k+j-1}{2}$.

To show that there is no other $\beta' \in \mathcal{O}_K$ with $v(\beta'^2 + w\beta' + w) > (r_k + j)v(\pi)$, suppose $\beta' = \pi^{r/2} + y$, and $y \in \mathcal{O}_K$. Then,

$$\beta'^2 + w\beta' + w = (\pi^{r/2} + y)^2 + w(\pi^{r/2} + y) + w = w\pi^j + 2\pi^{r/2}y + y^2 + wy.$$

It holds that $v(2\pi^{r/2}y) > v(2y) \geq v(wy)$, so the term with smallest valuation is among $w\pi^j$, y^2, and wy. Since j is odd, $v(y^2) \neq r_k + j$. If the valuation is greater than $v(w\pi^j) = (r_k + j)v(\pi)$, we must have

$$v(y^2) > (r_k + j)v(\pi), \text{ and } v(w\pi^j) = v(wy).$$

Therefore $v(y) = jv(\pi)$, thus $2j > r_k + j$ or $j > r_k$, a contradiction. This finishes the construction of K such that $d_k = \frac{r_k+j-1}{2}$, which ranges over all integers between $\frac{r_k}{2}$ and $r_k - 1$.

To complete the classification, suppose the uniformizer $\pi \in K$ is a root of the Eisenstein polynomial

$$f(x) = x^r - wg(x) = x^r - w(a_{r-1}x^{r-1} + \cdots + a_1x + a_0),\ a_i \in \mathcal{O}_{K_0},\ a_0 \in \mathcal{O}_{K_0}^\times.$$

Also, suppose $v(\beta^2+w\beta+w)$ attains its maximum at some β; from the reasoning above, we must have $v(\beta)=(r_k/2)v(\pi)$, so we assume

$$\beta\equiv\pi^{r/2}C(\pi)=\pi^{r/2}(c_0+c_1\pi+\dots c_{r/2-1}\pi^{r/2-1}) \bmod \pi^r,\ c_0,\dots,c_{r/2-1}\in\mathcal{O}_{K_0}.$$

By Lemma 2.3, $v(\beta^2+w\beta+w)\le 2r_kv(\pi)=2v(w)$, thus we can consider $\beta^2+w\beta+w \bmod w^2$:

$$\beta^2+w\beta+w\equiv\pi^rC^2(\pi)+w\pi^{r/2}C(\pi)+w\equiv w(C^2(\pi)g(\pi)+\pi^{r/2}C(\pi)+1) \bmod w^2.$$

The expression modulo w^2 remains unchanged when each coefficient a_i or c_i changes by an element of $w\mathcal{O}_{K_0}$, so we can assume these coefficients are taken in $\mathcal{O}_{K_0}/w\mathcal{O}_{K_0}=\mathbb{F}_q$, in particular $a_0\neq 0$. Among different polynomials $C(x)$, the largest possible $v(\beta^2+w\beta+w)$ corresponds to the smallest nonzero x^j term in $1+x^{r/2}C(x)+C^2(x)g(x)$, which is determined by Lemma 2.5. Suppose this is the x^l term. Then

$$\max(\beta^2+w\beta+w)=v(w\pi^l)=r_k+l,$$

and $d_k=\frac{r_k+l-1}{2}$.

□

As k varies, the widths d_k satisfy a more precise relationship.

Proposition 2.6. *In the setting of Theorem 2.1, suppose k_M is the largest integer such that* $\mathrm{SL}(2,K)$ *contains an element of order 2^{k_M}. Then for $3\le k\le k_M$, we have*

$$d_{k-1}=d_k+r_k.$$

In particular, if r_{k_M} is odd, then

$$d_k=r_k-\frac{r_{k_M}+1}{2}.$$

Proof. By the proofs of Lemmas 2.2 and 2.3,

$$2d_k+1=\max_{\alpha\in\mathcal{O}_K}2v_K(\alpha-\xi_{2^k})=\max_{\alpha\in\mathcal{O}_K}v_L(\alpha-\xi_{2^k}),$$

where $L=K(i)=K(\xi_{2^{k_M}})$. Fix a uniformizer π_L, and write $\xi_{2^k}=a_k+b_k\pi_L$, where $a_k,b_k\in K$. Therefore,

$$v_L(\alpha-\xi_{2^k})=\min\{v_L(\alpha-a_k),v_L(b_k\pi_L)\}=\min\{2v_K(\alpha-a_k),2v_K(b_k)+1\}.$$

Comparing parities, we see that $d_k=v_K(b_k)$. Applying the nontrivial automorphism $\sigma\in\mathrm{Aut}(L/K)$, we eliminate a_k and obtain

$$b_k=\frac{\xi_{2^k}-\xi_{2^k}^{-1}}{\pi_L-\sigma(\pi_L)}.$$

Combining this with the identity $\xi_{2^{k-1}}-\xi_{2^{k-1}}^{-1}=z_{2^k}(\xi_{2^k}-\xi_{2^k}^{-1})$ gives $b_{k-1}=z_{2^k}b_k$, so

$$d_{k-1}=d_k+v_K(z_{2^k})=d_k+r_k.$$

The formula when $2\nmid r_{k_M}$ then follows inductively from $d_{k_M}=\frac{r_{k_M}-1}{2}$.

□

Example 2.7 (cf. [7, Example before Proposition 4.4])**.** Let $k = 2$, $K_0 = \mathbb{Q}_2$, and $K = \mathbb{Q}_2(\sqrt{2})$, so $r_2 = e(K/\mathbb{Q}_2) = 2$. By Theorem 2.1, $r_2/2 = 1 \leq d_2 \leq 1 = r_2 - 1$, so the fixed subtree in $T_K = \mathrm{PSL}(2, K)/\mathrm{PSL}(2, \mathcal{O}_K)$ of an element of order $2^k = 4$ in $\mathrm{SL}(2, K)$ is the 1-neighborhood of an edge. In fact, the Eisenstein polynomial is $f(x) = x^2 - 2$, meaning $a_0 = 1$ and $a_1 = 0$. The equation for $j = 1$,

$$a_1 = \sqrt{a_0} \in \mathbb{F}_2$$

does not hold, thus our width formula implies $d_2 = \frac{r_2+j-1}{2} = 1$, in agreement with the bounds above.

By contrast, for elements of order 8 in $\mathrm{SL}(2, K)$ one has $k = 3$, $\mathbb{Q}_2(z_8) = \mathbb{Q}_2(\sqrt{2}) = K$, and $r_3 = e(K/K) = 1$, so $d_3 = \frac{r_3-1}{2} = 0$ and the fixed subtree is a single edge.

3. Vertex Stabilizers in $\mathrm{SL}(2, K)$-lattices

Lemma 3.1. *Let K be a finite extension of $\mathbb{Q}_2$. Then the following are equivalent:*

- *$|K/\mathbb{Q}_2|$ is even.*
- *There exist $x, y \in K$ such that $x^2 + y^2 = -1$.*
- *There exist $x, y \in \mathcal{O}_K$ such that $x^2 + y^2 = -1$.*

Proof. For $x, y \in \mathbb{Z}$, $x^2 + y^2 \equiv 0, 1, 2 \bmod 4$ thus $x^2 + y^2 \not\equiv -1 \bmod 4$. By Hensel's lemma, this implies that $x^2 + y^2 = -1$ has no solutions in the base field $\mathbb{Q}_2$. It immediately follows that the Hilbert symbol $(-1, -1)_{\mathbb{Q}_2} = -1$, and hence the Hasse invariant of the quaternion algebra is given by (see, e.g., [2], Exercise 10.4.4):

$$\mathrm{inv}_{\mathbb{Q}_2}\left(\frac{-1, -1}{\mathbb{Q}_2}\right) = [1/2] \in \mathbb{Q}/\mathbb{Z}.$$

For an extension $|K/\mathbb{Q}_2| = n$, the corresponding Hasse invariant follows the Brauer group restriction map, which corresponds to multiplication by the degree (see, e.g., [10], Chapter 13):

$$\mathrm{inv}_K\left(\frac{-1, -1}{K}\right) = |K/\mathbb{Q}_2| \cdot \mathrm{inv}_{\mathbb{Q}_2}\left(\frac{-1, -1}{\mathbb{Q}_2}\right) = [n/2],$$

which vanishes if and only if n is even. Therefore,

$$\exists x, y \in K,\ x^2 + y^2 = -1 \iff (-1, -1)_K = 1 \iff 2 \mid |K/\mathbb{Q}_2|.$$

In addition, if $x^2 + y^2 = -1$ holds in K, with $v(x) \leq v(y)$, $v(x) < 0$, then

$$\left(\frac{y}{x}\right)^2 + \left(\frac{1}{x}\right)^2 = -1,\ \frac{y}{x},\ \frac{1}{x} \in \mathcal{O}_K,$$

so a solution in $\mathcal{O}_K$ exists as well.

□

Proposition 3.2 (cf. [7, Proposition 3.3])**.** *Let K be a finite extension of $\mathbb{Q}_2$ with residue field $\mathbb{F}_q$.*

- *The group $2.A_4$ is isomorphic to a subgroup of $\mathrm{SL}(2, K)$ if and only if $|K/\mathbb{Q}_2|$ is even.*

- *The group* $2.S_4$ *is isomorphic to a subgroup of* $\mathrm{SL}(2,K)$ *if and only if* $z_8 \in K$.
- *The group* $2.A_5$ *is isomorphic to a subgroup of* $\mathrm{SL}(2,K)$ *if and only if* $q \equiv \pm 1 (\mathrm{mod} 5)$, *i.e., if* $z_5 \in K$.

Note that both conditions (2) and (3) imply (1): both $z_8 = \sqrt{2}$ and $z_5 = \frac{1+\sqrt{5}}{2}$ are quadratic over $\mathbb{Q}_2$, so extensions containing either element have even degree over $\mathbb{Q}_2$.

Proof. The same argument as in the proof of [7, Proposition 3.3] shows that $2.A_4$ or $2.A_5$ is contained in $\mathrm{SL}(2,K)$ if and only if A_4 or $A_5 < \mathrm{PGL}(2,K)$.

(a) We know $A_4 < \mathrm{PGL}(2,K)$ if and only if $\exists y, z \in K$ with $y^2 + z^2 = -1$. By Lemma 3.1, this is equivalent to the condition that $|K/\mathbb{Q}_2|$ is even.

(b) The subgroup $2.S_4 < \mathrm{SL}(2,K)$ if and only if $S_4 < \mathrm{PSL}(2,K)$. On the one hand, the only element of order 2 in $\mathrm{SL}(2,K)$ is $-I$ and projects to the identity in $\mathrm{PSL}(2,K)$, so the element in $\mathrm{PSL}(2,K)$ of order 4 necessitated by the inclusion must lift to an element of order 8 in $\mathrm{SL}(2,K)$, implying $\sqrt{2} = z_8 \in K$. On the other hand, when $\sqrt{2} \in K$, one can explicitly construct a subgroup $S_4 < \mathrm{PSL}(2, \mathbb{Q}_2(\sqrt{2})) \leq \mathrm{PSL}(2,K)$, which is explicitly treated in Proposition 3.10.

(c) $A_5 < \mathrm{PGL}(2,K)$ requires $\sqrt{5} \in K$, thus $z_5 = (\sqrt{5}+1)/2 \in K$. On the other hand, if $\sqrt{5} \in K$, then $|K/\mathbb{Q}_2| = |K/\mathbb{Q}_2(\sqrt{5})| \cdot |\mathbb{Q}_2(\sqrt{5})/\mathbb{Q}_2|$ is even, so $y^2 + z^2 = -1$ has a solution by (a). This implies $A_5 < \mathrm{PGL}(2,K)$ by [9].

□

3.1. **Dicyclic Subgroups of** $\mathrm{SL}(2,K)$.

Proposition 3.3 (cf. [7, Proposition 4.6]). *Let* $g \in \mathrm{SL}(2,K)$, $\mathrm{ord}(g) = 2^k$, $k \geq 2$, *and* $x \in \mathcal{V}(\mathfrak{C}_g)$.

- *If* $\mathfrak{C}_g$ *is a bi-infinite geodesic, there exists a subgroup* $A \cong C_{(q-1)2^k}.2 < \mathrm{SL}(2,K)_x$ *containing* g.
- *If* $\mathfrak{C}_g$ *is a single vertex, there exists a subgroup* $A \cong C_{(q+1)2^k}.2 < \mathrm{SL}(2,K)_x$ *containing* g.
- *Assume* $\mathfrak{C}_g$ *is a single edge. If* $|K/\mathbb{Q}_2|$ *is even, there exists a subgroup* $A \cong C_{2^k}.2 < \mathrm{SL}(2,K)_x$ *containing* g. *If* $|K/\mathbb{Q}_2|$ *is odd, there is no subgroup* $A \leq \mathrm{SL}(2,K)$ *with* $g \in A$ *and* $A \cong C_{2^k}.2$.

Proof. (1) By assumption, K contains the root of unity $\xi = \xi_{(q-1)2^k}$. The subgroup can be taken as

$$A = \left\langle \left(\begin{smallmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle \cong C_{(q-1)2^k}.2.$$

(2) According to the proof of [7, Theorem 4.2], it holds that

$$\alpha = \frac{\xi_{q+1}\xi_{2^k}^{-1} - \xi_{q+1}^{-1}\xi_{2^k}}{\xi_{q+1} - \xi_{q+1}^{-1}}, \ \beta = \frac{\xi_{2^k} - \xi_{2^k}^{-1}}{\xi_{q+1} - \xi_{q+1}^{-1}} \in K.$$

Therefore,

$$g = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha + \beta z_{q+1} \end{pmatrix},\ h = \begin{pmatrix} 0 & -1 \\ 1 & z_{q+1} \end{pmatrix}$$

commute and have orders 2^k and $q+1$, meaning $\langle g, h\rangle \cong C_{(q+1)2^k}$.

Finding an additional generator $\sigma \in \mathrm{SL}(2, K)$ requires

$$\sigma = \begin{pmatrix} x & xz_{q+1} + y \\ y & -x \end{pmatrix} \in \mathrm{SL}(2, K) \iff x^2 + y^2 + xyz_{q+1} = \|x + y\xi_{q+1}\|_{K(\xi_{q+1})/K} = -1.$$

Since $K(\xi_{q+1}) = K(\sqrt{-1})$ and $\sigma(\sqrt{-1}) = -\sqrt{-1}$, this is equivalent to finding $x, y \in K$ with $\|x + y\sqrt{-1}\|_{K(\sqrt{-1})/K} = x^2 + y^2 = -1$. As $K(\xi_{2^k})/K$ does not ramify, $\frac{1}{v_2(\pi_K)} = \frac{1}{v_2(\pi_{K(\xi_{2^k})})}$ which is even. Therefore $|K/\mathbb{Q}_2|$ is even, and by Lemma 3.1, $x^2 + y^2 = -1$ has solutions in K. This confirms the existence of $\langle g, h, \sigma\rangle \cong C_{(q+1)2^k}.2 < \mathrm{SL}(2, K)_x$.

(3) Let

$$g = \begin{pmatrix} 1 & z_{2^k} - 2 \\ 1 & z_{2^k} - 1 \end{pmatrix},$$

then $\langle g, \sigma\rangle \cong C_{2^k}.2$ if and only if σ takes the form

$$\sigma = \begin{pmatrix} x + y & (z-2)x \\ y & -x - y \end{pmatrix} \in \mathrm{SL}(2, K),$$

so the existence of σ is equivalent to the existence of a solution to

$$-1 = (x+y)^2 - (2 - z_{2^k})xy = x^2 + z_{2^k}xy + y^2 = \|x + y\xi_{2^k}\|_{K(\xi_{2^k})/K}.$$

Again, as $z_{2^k} \in K$, $K(\sqrt{-1}) = K(\xi_{2^k})$ and the existence of s is equivalent to the existence of $x, y \in K$ with $x^2 + y^2 = -1$, which is possible if and only if $|K/\mathbb{Q}_2|$ is even.

In all cases, since $\langle g\rangle$ is normal, the group A fixes at least one vertex of $\mathfrak{C}_g$.

□

Proposition 3.4 (cf. [7, Proposition 4.10])**.** *Assume $A \leq \mathrm{SL}(2, K)$ is a subgroup isomorphic to $C_{2^k}.2$.*

- *There exists a unique $\mathrm{GL}(2, K)$-conjugacy class of subgroups isomorphic to A.*
- *If r_k is even, T_A is an $r_k/2$-neighborhood of a vertex. If r_k is odd, T_A is an $(r_k - 1)/2$-neighborhood of an edge. For the generator g of the C_{2^k}-subgroup in A, $\mathfrak{C}_g \cap T_A$ consists of a single vertex if $\mathfrak{C}_g$ is either an infinite path or a vertex, and $\mathfrak{C}_g \cap T_A = \mathfrak{C}_g$ if it is a single edge.*

Proof. (1) This follows by the same argument as in the proof of [7, Proposition 4.10 (1)].

(2) (a) If $\xi_{2^k} \in K$, as in Proposition 3.3, we may take

$$g = \begin{pmatrix} \xi_{2^k} & 0 \\ 0 & \xi_{2^k}^{-1} \end{pmatrix},\ \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ A = \langle g, \sigma\rangle.$$

The axis of g joins 0 and $\infty \in \mathbf{P}^1(K)$, and that of σ joins $\pm\sqrt{-1} \in \mathbf{P}^1(K)$. By Theorem 1.1, the fixed trees $T_g = N(\mathfrak{C}_g, r_k)$, $T_\sigma = N(\mathfrak{C}_\sigma, r_2)$, where the cores $\mathfrak{C}_g, \mathfrak{C}_\sigma$ are the axes, with distance between them given by

$$d(\mathfrak{C}_g, \mathfrak{C}_\sigma) = \frac{v(2)}{v(\pi_K)} = r_2.$$

Since $k \geq 2$, $r_k = r_2/2^{k-2} \leq r_2$, we have

$$T_A = T_g \cap T_\sigma = N(\mathfrak{C}_g, r_k) \cap N(\mathfrak{C}_\sigma, r_2) = N(m, r_k/2),$$

where m is the midpoint of a path of length r_k contained in the path of length r_2 connecting $\mathfrak{C}_g$ and $\mathfrak{C}_\sigma$. It is either the $r_k/2$-neighborhood of a vertex or the $(r_k-1)/2$-neighborhood of an edge, depending on the parity of r_k.

(b.i) If $\xi_{2^k} \notin K$ and $K(\xi_{2^k})/K$ is unramified, we may take

$$g = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha + \beta z_{q+1} \end{pmatrix}, \ \sigma = \begin{pmatrix} x & xz_{q+1} + y \\ y & -x \end{pmatrix},$$

where

$$\alpha = \frac{\xi_{q+1}\xi_{2^k}^{-1} - \xi_{q+1}^{-1}\xi_{2^k}}{\xi_{q+1} - \xi_{q+1}^{-1}}, \ \beta = \frac{\xi_{2^k} - \xi_{2^k}^{-1}}{\xi_{q+1} - \xi_{q+1}^{-1}} \in K,$$

and $x, y \in K$, $x^2 + z_{q+1}xy + y^2 = -1$. Solutions x, y correspond to solutions of $x'^2 + y'^2 = -1$ by

$$x = x' - \frac{z_{q+1}}{\gamma}y', \ y = \frac{2}{\gamma}y',$$

where

$$\gamma = \frac{\xi_{q+1} - \xi_{q+1}^{-1}}{\sqrt{-1}} \in \mathcal{O}_K^\times,$$

so both $\frac{z}{\gamma}$ and $\frac{2}{\gamma}$ are in the ring of integers $\mathcal{O}_K$. By Lemma 3.1, $x'^2 + y'^2 = -1$ has integral solutions, so the corresponding x, y with $x^2 + z_{q+1}xy + y^2 = -1$ are also integral.

Since $v(\beta) = \frac{1}{2^{k-2}} = r_k v(\pi)$, g fixes a subpath of $(0, \infty)$ of length $2r_k$ centered at the origin; indeed the origin is the center of g. On the other hand, $v(z_{q+1}) = 0$, and since at least one of x and y has zero valuation, either y or $xz_{q+1} + y$ has zero valuation. This shows that the origin is in ∂T_σ. By a similar argument, $T_A = T_g \cap T_\sigma$ is either the $r_k/2$-neighborhood of a vertex or the $(r_k-1)/2$-neighborhood of an edge.

(b.ii) Since $K(\xi_{2^k}) = K(\sqrt{-1})$ is ramified over K, both $\mathfrak{C}_g$ and $\mathfrak{C}_\sigma$ are single edges. Thus $T_A = T_g \cap T_\sigma$ is the intersection of two edge-neighborhoods. On the vertex set, each edge-neighborhood is a ball centered at the midpoint of the edge, and the nonempty intersection of two balls in a tree is again a ball. Consequently, T_A is a neighborhood of either a vertex or an edge. It therefore suffices to show that its diameter is r_k.

On the one hand, the diameter of T_A is at least r_k: setting

$$g = \begin{pmatrix} 1 & z_{2^k} - 2 \\ 1 & z_{2^k} - 1 \end{pmatrix}, \ \sigma = \begin{pmatrix} x + y & (z-2)x \\ y & -x - y \end{pmatrix},$$

where $x, y \in \mathcal{O}_K$, we see that both matrices lie in the following vertex stabilizers,

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,K) \mid a,d \in \mathcal{O}_K,\ b \in \pi^i \mathcal{O}_K,\ c \in \pi^{-i}\mathcal{O}_K \right\},\ i = 0, \dots, v(z-2) = r_k,$$

that is, $T_g \cap T_\sigma = T_A$ contains a path of length r_k.

On the other hand, the diameter of T_A cannot be larger. If r_k is odd, then by Theorem 2.1, the diameter of T_g is already r_k. If r_k is even, up to a $\mathrm{GL}(2,K)$-conjugation we may assume to the contrary that T_g contains a subpath of $(0,\infty)$ of length $2d_k+1$, $\frac{r_k}{2} \le d_k \le r_k - 1$, and $T_\sigma \cap T_g$ contains a subpath of $(0,\infty)$ of length greater than r_k. It follows that

$$g = \begin{pmatrix} \alpha & \epsilon \\ \epsilon^{-1}(\alpha(z-\alpha)-1) & z-\alpha \end{pmatrix},\ \epsilon \in \mathcal{O}_K^\times,$$

with $v_K(\alpha(z-\alpha)-1) = 2d_k+1$, and the relationship $\sigma g = g^{-1}\sigma$ necessitates

$$\sigma = \begin{pmatrix} \frac{\epsilon^{-1}(\alpha(z-\alpha)-1)x+\epsilon y}{z-2\alpha} & x \\ y & -\frac{\epsilon^{-1}(\alpha(z-\alpha)-1)x+\epsilon y}{z-2\alpha} \end{pmatrix},\ v_K(xy) > r_k.$$

However, from the proof of Theorem 2.1 we see $v(\alpha-1) > 0$, so $v_K(2(1-\alpha)) > v_K(2) \ge r_k$; as $v_K(z-2) = r_k$, the denominator has valuation $v_K(z-2\alpha) = r_k$. The assumption that $T_\sigma \cap T_g$ contains a subpath of length $> r_k$ ultimately implies that the numerator has valuation $> r_k$, so $v(\det(\sigma)) > 0$, contradicting the assumption that $\sigma \in \mathrm{SL}(2,K)$.

□

Unlike in the odd-prime cases, the group $N_{\mathrm{GL}(2,K)}(A)$ need not act transitively on ∂T_A for $p = 2$:

Proposition 3.5. *In the setting of Proposition 3.4, each $N_{\mathrm{GL}(2,K)}(A)$-orbit of ∂T_A has at most 2 vertices if $k \ge 3$. If $k = 2$, then each $N_{\mathrm{GL}(2,K)}(A)$-orbit of ∂T_A has at most 6 vertices.*

Proof. Fix the dicyclic group presentation $A = \langle g, \sigma \mid g^{2^{k-1}} = \sigma^2 = -I,\ \sigma g \sigma^{-1} = g^{-1} \rangle$.

If $k \ge 3$, then for any $t \in \mathrm{GL}(2,K)$, since the t-adjoint map preserves the eigenvalues, we have

$$tgt^{-1} \in \{g, g^{-1}\},\ t\sigma t^{-1} = g^j \sigma,\ \exists j \in \mathbb{Z}.$$

On the other hand, the σ-adjoint map sends (g,σ) to (g^{-1},σ) while the g-adjoint map sends it to $(g, g^2\sigma)$. Therefore, the image of the map

$$N_{\mathrm{GL}(2,K)}(A) \to \mathrm{Out}(A)$$

is contained in a cyclic group of order 2, generated by $(g,\sigma) \mapsto (g, g\sigma)$. Since its kernel is $K^\times \times A$ and fixes T_A pointwise, each $N_{\mathrm{GL}(2,K)}(A)$-orbit of ∂T_A contains at most two vertices.

If $k = 2$, then A is isomorphic to the quaternion group Q_8, allowing a larger outer automorphism group $\mathrm{Out}(Q_8) \cong S_3$, generated by the permutations of $\{g, \sigma, g\sigma\}$. For the same reason, each $N_{\mathrm{GL}(2,K)}(A)$-orbit of ∂T_A contains at most 6 vertices.

The orbit size bounds above are attained. Indeed, if $k \geq 3$, then $t = I + g$ satisfies $tgt^{-1} = g$ and $t\sigma t^{-1} = g\sigma$. For $k = 2$, the elements $[I + g]$, $[I + \sigma] \in N_{\mathrm{GL}(2,K)}(A)/(K^\times \times A)$ serve as the generators of this copy of S_3.

□

As a result, an analogue of [7, Proposition 4.10, (2)] no longer holds: for $x \in \mathcal{V}(X)$, there are at least $|\partial T_A|/2$ (or $|\partial T_A|/6$ when $k = 2$) $\mathrm{GL}(2, K)_x$-conjugacy classes of groups $A^* \cong C_{2^k}.2$ with $x \in \partial T_{A^*}$ such that $A^* \cong A$. Consequently, the argument of [7, Lemma 4.7(3)], which plays a key role in their lattice constructions, does not carry over directly. Given a subgroup $A \cong C_{2^k}.2$ of $\mathrm{SL}(2, K)_x$ with $x \in \partial T_A$, the existence of a finite subgroup $B < \mathrm{SL}(2, K)_x$ containing a subgroup isomorphic to A does not in general guarantee the existence of a subgroup $B^* < \mathrm{SL}(2, K)_x$ containing A with $B^* \cong B$. This leads to the following question.

Question 3.6. *Let $A \cong C_{2^k}.2$ be a subgroup of $\mathrm{SL}(2, K)$. For each vertex $x \in \partial T_A$, which isomorphism types of finite proper supergroups $B > A$ can occur with $B < \mathrm{SL}(2, K)_x$?*

By the classification of finite subgroups [7, Theorem 3.1], any such B is isomorphic to $C_{s\cdot 2^l}.2$, where s is odd, $l \geq k$, and $s \mid q - 1$ or $s \mid q + 1$; to $2.A_4$ or $2.A_5$ when $k = 2$; or to $2.S_4$ when $k \leq 3$. The following proposition handles the case when B is dicyclic.

Proposition 3.7. *Fix a subgroup $A < \mathrm{SL}(2, K)$ with presentation $A = \langle g, \sigma \mid g^{2^{k-1}} = \sigma^2 = -I,\ \sigma g \sigma^{-1} = g^{-1} \rangle \cong C_{2^k}.2$. In the following, consider only integers $l \geq k$ and odd integers $s \geq 1$ such that $C_{s\cdot 2^l}.2$ is isomorphic to a subgroup of $\mathrm{SL}(2, K)$.*

If $k \geq 3$, then a vertex $x \in \partial T_A$ allows a supergroup $B > A$, $B < \mathrm{SL}(2, K)_x$ and $B \cong C_{2^l}.2$ if and only if $d(x, \mathfrak{C}_g) \leq d_l$, where d_l is the width defined in Theorem 1.1. For $s > 1$, a vertex $x \in \partial T_A$ allows a supergroup $B > A$, $B < \mathrm{SL}(2, K)_x$ and $B \cong C_{s\cdot 2^l}.2$ if and only if $d(x, \mathfrak{C}_g) = 0$.

If $k = 2$, a similar conclusion holds with $\mathfrak{C}_g$ being replaced by $\mathfrak{C}_g \cup \mathfrak{C}_\sigma \cup \mathfrak{C}_{g\sigma}$.

Proof. To be continued...

□

To discuss the cases when B is binary polyhedral, we need the following lemma describing the structure of T_{Q_8} and $T_{Q_{16}}$.

Lemma 3.8. *Fix the presentation $A = \langle g, \sigma \mid g^{2^{k-1}} = \sigma^2 = -I,\ \sigma g \sigma^{-1} = g^{-1} \rangle \cong C_{2^k}.2$.*

- *If $k = 2$ and a subgroup of $\mathrm{SL}(2, K)$ is isomorphic to $2.A_4$, then there exists a unique element $q \in \mathrm{SL}(2, K)$, $\mathrm{ord}(q) = 3$, such that $qgq^{-1} = \sigma$ and $q^{-1}gq = g\sigma$. Moreover, T_q is either the central vertex of T_A or a bi-infinite geodesic passing through the center of T_A.*
- *If $k = 3$ and a subgroup of $\mathrm{SL}(2, K)$ is isomorphic to $2.S_4$, then there exist unique elements $q_i \in \mathrm{SL}(2, K)$, $\mathrm{ord}(q_i) = 3$, $i = 1, 2$, such that $q_i g^2 q_i^{-1} = g^{i-1}\sigma$ and $q_i^{-1} g^2 q_i = g^{i+1}\sigma$. Moreover, for $i = 1, 2$, T_{q_i} meets T_A exactly at a vertex in ∂T_A, and the distance between these vertices is r_3.*

Proof. To be continued...

□

Proposition 3.9. *If $k = 2$, then $x \in \partial T_A$ allows a supergroup $A < B < \mathrm{SL}(2,K)_x$ with $B \cong 2.A_4$ or $B \cong 2.A_5$ if and only if T_q is an infinite path and $x \in T_q \cap \partial T_A$. B cannot be isomorphic to $2.S_4$.*

If $k = 3$, then $x \in \partial T_A$ allows a supergroup $A < B < \mathrm{SL}(2,K)_x$ with $B \cong 2.S_4$ if and only if $x \in T_{q_i} \cap \partial T_A$, $q = 1, 2$.

Proof. To be continued...

□

3.2. **Binary Polyhedral Subgroups.**

Proposition 3.10. *Let $g \in \mathrm{SL}(2,K)$ satisfy $\mathrm{ord}(g) = 4$, and suppose that $|K/\mathbb{Q}_2|$ is even.*

- *There exists $x \in \mathcal{V}(T_g)$ such that there is a subgroup $A \cong 2.A_4 < \mathrm{SL}(2,K)_x$ containing g.*
- *If $\sqrt{2} \in K$, there also exists a subgroup $A \cong 2.S_4 < \mathrm{SL}(2,K)_x$ containing g.*

Proof. Lemma 3.1 implies the existence of $x, y \in K$ such that $x^2 + y^2 = -1$. Define the matrices:

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; q = \frac{1}{2}\begin{pmatrix} 1-x+y & 1+x+y \\ -1+x+y & 1+x-y \end{pmatrix},$$

so

$$\det(g) = 1, \; \det(q) = \frac{1}{4}((1-(x-y)^2) - ((x+y)^2 - 1)) = \frac{1}{4}(2 - 2(x^2+y^2)) = 1,$$

and

$$\mathrm{tr}(g) = 0, \; \mathrm{tr}(q) = 1,$$

so $\mathrm{ord}(g) = 2$ and $\mathrm{ord}(q) = 3$. At the same time,

$$gq = \frac{1}{2}\begin{pmatrix} -1+x+y & 1+x-y \\ -1+x-y & -1-x-y \end{pmatrix}, \; \mathrm{tr}(gq) = -1,$$

so $\mathrm{ord}(gq) = 3$. Therefore, the group generated by $g, q \in \mathrm{PSL}(2,K)$ is

$$\langle g, q \mid g^2, q^3, (gq)^3 \rangle \cong A_4.$$

If $\sqrt{2} \in K$, we can take

$$x = 2 + \sqrt{\frac{-6}{5}}, \; y = 1 - 2\sqrt{\frac{-6}{5}} \in K,$$

as $-\frac{3}{5} \equiv 1 (\mathrm{mod}\ 8)$.

In addition, we define

$$p = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

so $p^2 = g$ and $\mathrm{tr}(p) = \sqrt{2}$, and $\mathrm{ord}(p) = 4$. At the same time,

$$pq = \frac{\sqrt{2}}{2}\begin{pmatrix} y & x+1 \\ x-1 & -y \end{pmatrix},\ \mathrm{tr}(pq) = 0,$$

so $\mathrm{ord}(pq) = 2$. Therefore, the group generated by $p, q \in \mathrm{PSL}(2, \mathbb{Q}_2(\sqrt{2}))$ is

$$\langle p, q \mid p^4, q^3, (pq)^2 \rangle \cong S_4.$$

These groups lift to $2.A_4$ and $2.S_4$ in $\mathrm{SL}(2, K)$.

□

Proposition 3.11. *In the setting of Proposition 3.10, suppose x is taken such that $d(x, \mathfrak{C}_g)$ is minimized.*

- *If $\mathfrak{C}_g$ is either a bi-infinite geodesic or a single vertex, then $d(x, \mathfrak{C}_g) = r_2/2$.*
- *If $\mathfrak{C}_g$ is an edge and r_2 is even, then $d(x, \mathfrak{C}_g) = d_2 - r_2/2$.*
- *If $\mathfrak{C}_g$ is an edge and r_2 is odd, then $x \in \mathfrak{C}_g$.*

Proof. Consider the non-trivial elements in the subgroup $Q \triangleleft A$, $Q \cong Q_8$, namely

$$g = p^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ qgq^{-1} = \begin{pmatrix} -x & y \\ y & x \end{pmatrix},\ q^{-1}gq = \begin{pmatrix} y & x \\ x & -y \end{pmatrix}.$$

We notice that the intersection of $T_g = N(\mathfrak{C}_g, d_2)$, $T_{qgq^{-1}} = N(\mathfrak{C}_{qgq^{-1}}, d_2)$ and $T_{q^{-1}gq} = N(\mathfrak{C}_{q^{-1}gq}, d_2)$ coincides with T_Q, which is a ball of diameter r_2 by Proposition 3.4.

Suppose first that $\mathfrak{C}_g$ is a bi-infinite geodesic or a single vertex. The three cores $\mathfrak{C}_g$, $\mathfrak{C}_{qgq^{-1}}$ and $\mathfrak{C}_{q^{-1}gq}$ are cyclically permuted by q, and since $d_2 = r_2$ in these cases, they cannot have a common vertex. By symmetry of these cores, they are connected by a tripod. Since A permutes these cores, the tripod center x is fixed by A, and is the vertex in T_A closest to $\mathfrak{C}_g$. Using the diameter of T_Q and the width of T_g, it is straightforward to see that $d(x, \mathfrak{C}_g) = r_2/2$.

Now we consider the case when $\mathfrak{C}_g$ is an edge. If the three edges coincide, then the diameter of $T_Q = N(\mathfrak{C}_g, d_2)$ is $r_2 = 2d_2 + 1$, which is odd. Since $Q \triangleleft A$, the group A preserves the central edge $\mathfrak{C}_g$ so fixes it pointwise. Therefore, x is one of the vertices of $\mathfrak{C}_g$.

If the three edges are distinct, then their induced subtree is a tripod. As $Q \triangleleft A$, A permutes the three core edges and hence fixes their tripod center x. Thus, it is the vertex in T_A closest to $\mathfrak{C}_g$. Since T_Q is the intersection of T_g with its two images under q, x is the center of T_Q. Using the diameters of T_g and T_Q, it is straightforward to see that r_2 is even, and $d(x, \mathfrak{C}_g) = d_2 - r_2/2$.

□

Unlike in the odd-p cases, the binary polyhedral group $2.A_4$ may occur as an edge group for minimal covolume lattices of $\mathrm{SL}(2, K)$ with $K/\mathbb{Q}_2$. For example, the construction in Section 4 gives a lattice

$$\Gamma \cong 2.A_5 *_{2.A_4} 2.A_5,\ \Gamma < \mathrm{SL}(2, \mathbb{Q}(\xi_3)),$$

with covolume $\frac{1}{120}$. For this reason, we similarly discuss the fixed subtree T_A for $A \cong 2.A_4$ and how supergroups $B > A$ can be attached at vertices $x \in \partial T_A$.

Corollary 3.12. *Assume $A \leq \mathrm{SL}(2,K)$ is a subgroup isomorphic to $2.A_4$, and let Q be its unique normal subgroup isomorphic to Q_8.*

- *There exists a unique $\mathrm{GL}(2,K)$-conjugacy class of subgroups isomorphic to A.*
- *If $2 \mid f$, then T_A is a path of length r_2 sharing its center with T_Q. If $2 \nmid f$, then T_A consists of the central vertex of T_Q.*
- *If $2 \mid f$, then for each $x \in \partial T_A$, there exists a supergroup $A < B < \mathrm{SL}(2,K)_x$ with $B \cong 2.A_5$, whereas no such supergroup is isomorphic to $2.S_4$.*

Proof. To be continued...

□

3.3. **Restrictions on Vertex Stabilizers.**

Proposition 3.13 (cf. [7, Proposition 3.4])**.** *Let $g \in \mathrm{SL}(2,\mathcal{O}_K)$ be an element of order 2^k, $k \geq 2$, such that*

$$g \notin \ker(\tau :\ \mathrm{SL}(2,\mathcal{O}_K) \to \mathrm{SL}(2,\mathbb{F}_q)).$$

Then for a finite subgroup $S < \mathrm{SL}(2,\mathcal{O}_K)$[1] containing g, one of the following holds:

- *The group $C_{\mathrm{SL}(2,\mathcal{O})}(g) \cap S$ is a cyclic group of order 2^k and has index at most 2 in S.*
- *$k = 2$, $C_{\mathrm{SL}(2,\mathcal{O})}(g) \cap S = \langle g \rangle = C_4$, and g is not contained in the index 2 cyclic subgroup of S described in [7, Theorem 3.1].*
- *$k \in \{2,3\}$ and $S \cong 2.A_4$, $2.S_4$, or $2.A_5$.*

Proof. When S is not isomorphic to $2.A_4$, $2.S_4$, or $2.A_5$, it is contained in a maximal subgroup of type (a) or (b) as described in [7, Theorem 3.1].

Note that the maximal subgroup $\mu_K.2$ or $\mu^*_{E/K}.2$ is a dicyclic group. When g is not contained in the cyclic subgroup μ_K or $\mu^*_{E/K}$, it holds that $\mathrm{ord}(g) = 4$, and its centralizer in the dicyclic group is the order-four subgroup generated by it, and so is its intersection with S.

In other cases, the centralizer is the cyclic group μ_K or $\mu^*_{E/K}$, thus its intersection with S is cyclic with index 1 or 2 in S, depending on whether S contains elements in $\mu_K.2 - \mu_K$. In these two situations, the largest normal pro-2 subgroup $O_2(S)$ could be either cyclic or dicyclic. The condition $g \notin \ker(\tau)$ implies g is not a square in $O_2(S)$. If $O_2(S)$ is cyclic, this implies $\langle g \rangle = O_2(S)$, as in [7]. If $O_2(S)$ is dicyclic, g lies in its index-2 core by our assumption, and we deduce that $\langle g \rangle$ equals this cyclic subgroup similarly. Since $\mathbb{F}_q$ has characteristic 2,

[1] [7, Proposition 3.4] appears to have a typo here: their statement with the assumption $S < \mathrm{SL}(2,K)$ does not hold; in fact, this assumption is inconsistent with the context and is not needed in this generality later.

$C_{\mathrm{SL}(2,\mathbb{F}_q)}(\tau(g)) \cong (C_2)^f$, implying that the centralizer $C_S(g)$ is also a 2-group. This centralizer has order at least $\mathrm{ord}(g) = 2^k$, and the maximality of $O_2(S)$ confirms that $|C_S(g)| = 2^k$.

□

The next proposition concerns vertex stabilizers away from the boundary of the fixed subtree:

Proposition 3.14 (cf. [7, Proposition 4.5 (b) and (c)])**.** *Let $g \in \mathrm{SL}(2,K)$ be an element of order 2^k, $k \geq 2$, with fixed tree T_g and center $\mathfrak{C}_g$. Let Γ be any lattice containing g, and $2^r \mid\mid |\mathrm{Tor}(C_{\mathrm{SL}(2,K)}(g))|$. Let x be a vertex in $T_g - \partial T_g$.*

(a) If x is in $\mathfrak{C}_g$:

- *If $\mathfrak{C}_g$ is a bi-infinite geodesic, one has $\Gamma_x \leq C_{(q-1)2^r}.2$.*
- *If $\mathfrak{C}_g$ is a single vertex, one has $\Gamma_x \leq C_{(q+1)2^r}.2$.*
- *If $\mathfrak{C}_g$ is a single edge, one has $\Gamma_x \leq C_{2^r}.2$, or $k = 2$ and $\Gamma_x = 2.A_4$ or $\Gamma_x = 2.S_4$.*

(b) If x is not in $\mathfrak{C}_g$, one has either $\Gamma_x \leq C_{2^r}.2$, or $k = 2$ and $\Gamma_x = 2.A_4$ or $\Gamma_x = 2.S_4$.

Proof. (a) Since we assume that $x \notin \partial T_g$ (which is automatic in cases (i) and (ii)), g fixes the star of x. Realizing the full stabilizer of x as $\mathrm{SL}(2,\mathcal{O}_K)$, we have $g \in \Gamma_{\mathrm{st}(x)} = \Gamma_x \cap \ker(\tau)$. Note that $\Gamma_{\mathrm{st}(x)} \lhd \Gamma_x$ and $\ker(\tau)$ is pro-2, thus Γ_x contains a normal 2-subgroup containing g.

By [7, Theorem 3.1], Γ_x is contained in either a dicyclic group (cases (a) and (b)), or it is an exceptional group $2.A_4$, $2.S_4$ or $2.A_5$ (cases (c) to (e)). The first two contain a normal 2-subgroup isomorphic to Q_8, while $2.A_5$ only contains $\{\pm I\}$, which is not possible here. If $\mathrm{ord}(g) \geq 8$, then the exceptional cases cannot hold, and an argument similar to that in [7, Proposition 4.5] implies the assertions in cases (i) and (ii). If $\mathrm{ord}(g) = 4$, Γ_x does not contain $2.A_4$ for $x \in \mathfrak{C}_g$, which is proved in Proposition 3.10.

In case (iii), while $\Gamma_x \leq C_{(q+1)2^r}.2$ (or $2.A_4$ or $2.S_4$) holds, the same argument as in [7, Proposition 4.5] shows that, for the eigenvalues α and $\sigma(\alpha)$ of elements in Γ_x, where σ generates $\mathrm{Gal}(K(\xi_{2^r})/K)$, α lies in the pro-2 group $(1 + \pi_L\mathcal{O}_L) = (-1 + \pi_L\mathcal{O}_L)$. It follows that $\Gamma_x \leq C_{2^r}.2$, or $\Gamma_x = 2.A_4$ or $2.S_4$ when $k = 2$.

(b) Take $h \in \langle g \rangle$ with $\mathrm{ord}(h) = 4$. Then h also fixes the star $\mathrm{st}(x)$ and makes $O_2(\Gamma)$ larger than $\{\pm I\}$. If $\mathrm{ord}(g) \geq 8$, the exceptional groups $2.A_4$ and $2.S_4$ are ruled out, and the same argument as in [7, Proposition 4.5] implies $|\Gamma_x : C_{\Gamma_x}(h)| = 2$. In particular, $O_2(C_{\mathrm{SL}(2,K)}(h)) \cong C_{2^r}$, while $O_{2'}(C_{\Gamma_x}(h)) = \{1\}$ as in the proof of [7, Proposition 4.5]. Putting these together gives the desired proposition.

If $\mathrm{ord}(g) = 4$, the exceptional situation that $\Gamma_x = 2.A_4$ or $2.S_4$ remains possible; indeed, it occurs exactly when $d(x, \mathfrak{C}_g) = r_2/2$ according to Proposition 3.11.

□

4. Construction of Lattices for $\mathbb{Q}_2$-extensions

Proposition 4.1 (cf. [7, Proposition 6.1]). *Let $R \cong 2.A_4$, $2.S_4$ or $2.A_5$ be a finite subgroup of $\mathrm{SL}(2,K)$ with residue field $\mathbb{F}_q$, $q = 2^f$. Suppose R is contained in $\mathrm{SL}(2,K)_v$ for some $v \in \mathcal{V}(X)$. Let $\rho : \mathrm{SL}(2,K)_v \cong \mathrm{SL}(2,\mathcal{O}_K) \to \mathrm{PSL}(2,\mathbb{F}_q)$ be the $\mathrm{mod}\pi$ projection.*

- *Let $R \cong 2.A_4$. Then $\rho(R) \cong C_3$ or A_4. If $\rho(R) \cong C_3$, then*

$$|\mathrm{st}(v)^\circ/R| = \begin{cases} (2^f+1)/3, & 2 \nmid f, \\ (2^f+5)/3, & 2 \mid f. \end{cases}$$

  If $\rho(R) \cong A_4$, then $2 \mid f$, and

$$|\mathrm{st}(v)^\circ/R| = (2^f+20)/12.$$

  In particular, $|\mathrm{st}(v)^\circ/R| = 1$ if and only if $q = 2$, $|\mathrm{st}(v)^\circ/R| = 2$ only if $q = 4$ and $\rho(R) \cong A_4$, and $|\mathrm{st}(v)^\circ/R| = 3$ if and only if $q = 4, 8$ and $\rho(R) \cong C_3$, or $q = 16$ and $\rho(R) \cong A_4$.

- *Let $R \cong 2.S_4$. Then $\rho(R) \cong S_3$, and*

$$|\mathrm{st}(v)^\circ/R| = \begin{cases} (2^f+4)/6, & 2 \nmid f, \\ (2^f+8)/6, & 2 \mid f. \end{cases}$$

  In particular, $|\mathrm{st}(v)^\circ/R| = 1$ if and only if $q = 2$, and $|\mathrm{st}(v)^\circ/R| = 2$ if and only if $q = 4, 8$.

- *Let $R \cong 2.A_5$. Then $2 \mid f$, $\rho(R) \cong A_5$, and*

$$|\mathrm{st}(v)^\circ/R| = \begin{cases} (2^f+56)/60, & 4 \nmid f, \\ (2^f+104)/60, & 4 \mid f. \end{cases}$$

  In particular, $|\mathrm{st}(v)^\circ/R| = 1$ if and only if $q = 4$, and $|\mathrm{st}(v)^\circ/R| = 2$ if and only if $q = 16, 64$.

Proof. First we determine the image $\rho(R)$. Since $\ker(\rho\,|_R)$ contains $-I$ and is normal in R, it is either isomorphic to C_2, or isomorphic to Q_8 when $R \cong 2.A_4$ or $2.S_4$. If $R \cong 2.A_4$, $\rho(R) \cong C_3$ or A_4. Since $\mathrm{PSL}(2,\mathbb{F}_{2^f})$ does not contain an element of order 4, it does not contain a subgroup isomorphic to S_4. Thus, if $R \cong 2.S_4$, we must have $\rho(R) \cong S_3$. Finally, $\rho(R) \cong A_5$ if $R \cong 2.A_5$.

For $g \in \mathrm{PSL}(2,\mathbb{F}_q)$, the number of leaves it fixes in $\mathrm{st}(v)^\circ$ is $(q+1)$ if $g = \mathrm{id}$, 1 if $\mathrm{ord}(g) = 2$, 0 if $\mathrm{ord}(g) \mid q+1$, and 2 if $\mathrm{ord}(g) \mid q-1$. Counting the number of elements in C_3, S_3, A_4 and A_5 of each order and applying Burnside's lemma give the formulas for the numbers of orbits given above.

□

Examples of Lattices in $\mathrm{SL}(2,K)$ for finite extensions $K/\mathbb{Q}_2$. To be continued...

5. Proof of Minimal Covolume Results

Proposition 5.1 (cf. [7, Proposition C]). *For a finite extension $K/\mathbb{Q}_2$ with residue field of order $q = 2^f$, the minimal covolume among cocompact lattices $G < \mathrm{SL}(2, K)$ without elements of order 4 is $\frac{1}{2(q+1)}$.*

The proof follows directly from [7, Example (6.1) and proof of Proposition C].

Strengthen this to exclude only elements of order 8?

Lemma 5.2 (cf. [7, Lemma 7.1 and 7.2]). *Let $q = 2, 4, 8$ or 16, and let Γ be a lattice of minimal covolume containing an element g of order 8. To be continued...*

Proof. To be continued...

□

Theorem 5.3 (cf. [7, Theorem 7.3, 7.4, and 7.5]). *Let $p = 2$ and let G be a lattice of minimal covolume containing an element g of order 8. To be continued...*

Proof. To be continued...

□

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